

# ORDERABLE GROUPS

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*Para Antonia, por todo*

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## Resumen:

En este trabajo estudiamos grupos ordenables. Ponemos especial énfasis en órdenes de tipo Conrad.

En el Capítulo 1 recordamos algunos resultados y definiciones básicos. También damos una nueva caracterización de la propiedad de Conrad.

En el Capítulo 2 usamos dicha nueva caracterización para obtener una clasificación de los grupos que admiten solo una cantidad finita de órdenes Conrad §2.1. Con esta clasificación en la mano, somos capaces de mostrar que el espacio de órdenes Conrad es finito, o bien no contiene órdenes aislados §2.2. Finalmente, la nueva caracterización de órdenes Conrad nos permite dar un teorema de estructura para el espacio de órdenes a izquierda, esto tras analizar la posibilidad de aproximar un orden por sus conjugados §2.3.

En el Capítulo 3, mostramos que, para grupos que admiten solo una cantidad finita de órdenes Conrad, es equivalente tener un orden a izquierda aislado que tener finitos órdenes a izquierda.

En el Capítulo 4 probamos que el espacio de órdenes a izquierda del grupo libre a dos o mas generadores, tiene una órbita densa bajo la acción natural de éste grupo en dicho espacio. Esto resulta en una nueva demostración del hecho que el espacio de órdenes a izquierda del grupo libre en dos o mas generadores no tiene órdenes aislados.

En el Capítulo 5 describimos el espacio de bi-órdenes del grupo de Thompson F. Mostramos que este espacio está compuesto de 8 puntos aislados junto con 4 copias canónicas del conjunto de Cantor.

## Abstract:

In this work we study orderable groups. We put special attention to Conradian orderings.

In Chapter 1 we give the basic background and notations. We also give a new characterization of the Conrad property for orderings.

In Chapter 2, we use the new characterization of the Conradian property to give a classification of groups admitting finitely many Conradian orderings §2.1. Using this classification we deduce a structure theorem for the space of Conradian orderings §2.2. In addition, we are able to give a structure theorem for the space of left-orderings on a group by studying the possibility of approximating a given ordering by its conjugates §2.3.

In Chapter 3 we show that, for groups having finitely many Conradian orderings, having an isolated left-ordering is equivalent to having only finitely many left-orderings.

In Chapter 4, we prove that the space of left-orderings of the free group on  $n \geq 2$  generators have a dense orbit under the natural action of the free group on it. This gives a new proof of the fact that the space of left-orderings of the free group in at least two generators have no isolated point.

In Chapter 5, we describe the space of bi-orderings of the Thompson's group  $F$ . We show that this space contains eight isolated points together with four canonical copies of the Cantor set.

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# Chapter 1

## Introduction

A *left-ordered group*  $G$  is a group  $G$  with a (total) order relation  $\preceq$  that is invariant under left multiplication. That is,  $f \prec g$  implies  $hf \prec hg$  for any  $f, g, h$  in  $G$ . If, in addition, we have that  $f \prec g$  implies  $hfh^{-1} \prec hgh^{-1}$  for all  $f, g, h$  in  $G$ , then we say that  $G$  is *bi-ordered* or that  $G$  has a bi-invariant ordering. We will use the term *ordered* when there is no harm of ambiguity (*e.g.* when  $G$  is Abelian).

The theory of orderable groups is a venerable subject of mathematics whose starting point are the works of R. Dedekind and O. Hölder at the end of XIX century and at the beginning of XX century, respectively. Dedekind characterizes the real numbers as a complete ordered Abelian group, while Hölder proves that any *Archimedean*<sup>1</sup> Abelian ordered group is order isomorphic to a subgroup of the additive real numbers with the standard ordering; see [18] or [14] for a modern version of this.

Besides the two different kinds of orderings described above, there is a third type which will be shown to be of great importance in this work. These are left-orderings satisfying

$$f \succ id \text{ and } g \succ id \Rightarrow fg^n \succ g \text{ for some } n \in \mathbb{N} = \{1, 2, \dots\}.$$

These so-called *Conradian* orderings (or  $\mathcal{C}$ -orderings) were introduced, in the late fifties, by P. Conrad in his seminal work [10]. There, Conrad shows that the above condition on a left-ordered group is equivalent to the fact that the conclusion of Hölder's theorem holds "locally" (see (4) below). Since their introduction, Conradian orderings have played a fundamental role in the theory of left-orderable groups; see, for instance, [2, 23, 27, 31, 35, 41]. Actually, for some time, it was an open question whether any left-orderable group admits a Conradian ordering. To the best of our knowledge, the first example of a left-orderable group admitting no  $\mathcal{C}$ -ordering appears in [40], but, apparently, this was not widely known (among people mostly interested in ordered groups) until [1] appeared.

For the statement of Conrad's theorem recall that, in a left-ordered group  $(G, \preceq)$ , a subset  $S$  is *convex* if whenever  $f_1 \prec h \prec f_2$  for some  $f_1, f_2$  in  $S$ , we have  $h \in S$ . As it is easy to check, the family of convex subgroups is linearly ordered under inclusion [15, 20, 35]. In particular, (arbitrary) unions and intersections of convex subgroups is also a convex subgroup. Therefore, for every  $g \in G$ , there exists  $G_g$  (resp.  $G^g$ ), the largest (resp. smaller) convex subgroup that does not contain  $g$  (resp. does contain  $g$ ). The inclusion  $G_g \subset G^g$  is typically referred to as the  $\preceq$ -*convex jump* associated to  $g$ .

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<sup>1</sup>An ordering is Archimedean if for any  $a \prec b, a \neq id$ , there exists  $n \in \mathbb{Z}$  such that  $b \prec a^n$ .

**Theorem 1.0.1 (Conrad).** *A left-ordering  $\preceq$  on a group  $G$  is said to be Conradian if the following four equivalent properties hold:*

- (1) *For all  $f \succ id$  and  $g \succ id$  (for all positive  $f, g$ , for short), we have  $fg^n \succ g$  for some  $n \in \mathbb{N}$ .*
- (2) *If  $1 \prec g \prec f$ , then  $g^{-1}f^n g \succ f$  for some  $n \in \mathbb{N}$ .*
- (3) *For all positive  $g \in G$ , the set  $S_g = \{f \in G \mid f^n \prec g, \text{ for all } n \in \mathbb{Z}\}$  is a convex subgroup.*
- (4) *For every  $g$ , we have that  $G_g$  is normal in  $G^g$ , and there exists a non-decreasing group homomorphism (to be referred to as the Conrad homomorphism)  $\tau_g^g : G^g \rightarrow \mathbb{R}$  whose kernel coincides with  $G_g$ . Moreover, this homomorphism is unique up to multiplication by a positive real number.*

About a decade ago, a new tool for studying left-orderable groups, the so-called *space of left-orderings* of a left-orderable group, was introduced by Ghys and, independently, by Sikora [37]. Roughly, the space of left-orderings of a group  $G$  is the set of all left-orderings of  $G$ , where we declare two left-orderings to be “close” if they coincide on a large finite subset of  $G$ . This object turns out to be a Hausdorff, totally disconnected and compact topological space on which  $G$  acts by conjugacy: given  $\preceq$ , a left-ordering on  $G$ , and  $f, g \in G$ , we define  $\preceq_f$  by  $id \preceq_f g$  if and only if  $id \preceq fgf^{-1}$ ; see §1.3 for details. Although this object appears for the first time in the literature in [37], it was in [23] and specially [27], that the full strength of this object was stressed. In [23], Linnell put to great use the compactness of the space of left-orderings to show that if a group admits infinitely many left-orderings, then it admits uncountably infinitely many. On the other hand, in [27], Morris-Witte squeezes the dynamics of a group acting on its space of left-orderings, to show that an amenable, left-orderable group must admit a Conradian ordering.

As it was noticed in [21, 31], in (1) and (2) above one may actually take  $n=2$ . The topological counterpart of this is the fact that the set of  $\mathcal{C}$ -orderings (of a given group) is compact when it is endowed with the natural topology; see §1.3. This leads, for instance, to a new and short proof of the fact, first proved by Brodskii in [4], that *locally indicable*<sup>2</sup> groups are  $\mathcal{C}$ -orderable [31, Proposition 3.11]; see also [20, Corollary 3.2.2]. In particular, the class of  $\mathcal{C}$ -orderable groups contains the class of torsion-free one-relator groups; see [4]. Note that, from (4) above, the converse to this result also holds, that is,  $\mathcal{C}$ -orderable groups are locally indicable. Indeed, in a finitely generated,  $\mathcal{C}$ -ordered group  $(G, \preceq)$ , the homomorphism  $\tau_g^g : G \rightarrow \mathbb{R}$ , where  $g \in G$  is the “largest” element in the generating set of  $G$ , is nontrivial.

One last important ingredient, fundamental for our work, is the dynamical content of the Conradian property for left-orderings revealed by Navas in [31]. There, Navas shows that a left-ordering on a countable group is Conradian if and only if some natural action on the real line, the so-called *dynamical realization of a left-ordering* (which we trace back to [14]), has no *crossings*; see Proposition 1.4.2 for the definition of the action and §1.4.1 for the definition of crossings.

What the concept of crossings is encoding, is the fact that the action of left translation of  $G$  on itself has some sort of well-behaved “levels” structure, which puts great constraint to the dynamics on the cosets of convex subgroups. Nevertheless, as illustrated in §1.2, Conradian orderings, unlike bi-invariant orderings, shares many nice properties with left-orderings, especially those related to possible modifications. It is this mixture between rigidity and flexibility what makes Conradian orderings a good stand point in the study of the more general left-orderings, and also, what makes them a very nice object of study.

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<sup>2</sup>A group  $\Gamma$  is *locally indicable* if for any nontrivial finitely generated subgroup  $H$ , there exists a nontrivial group homomorphism from  $H$  to the group of real numbers under addition.



## 1.1 Description of the main results

Our first main result is the generalization for uncountable groups of Navas dynamical characterization of Conradian orderings. To do this, we had to understand the concept of crossings in an intrinsic way. For us, a crossing for an action by order preserving bijections of a group  $G$  on a totally ordered space  $(\Omega, \leq)$ , is a 5-uple  $(f, g, u, v, w)$ , where  $f, g$  (resp.  $u, v, w$ ) belong to  $G$  (resp.  $\Omega$ ), that satisfies:

- (i)  $u < w < v$ .
- (ii) For every  $n \in \mathbb{N}$ , we have  $g^n u < v$  and  $f^n v > u$ .
- (iii) There exist  $M, N$  in  $\mathbb{N}$  such that  $f^N v < w < g^M u$ .

In §1.4.1 we show

**Theorem A.** *A left-ordering  $\preceq$  on a group  $G$  is Conradian if and only if the action by left translation on itself contains no crossings.*

We point out that, besides the four equivalences of the Conrad property given in Theorem 1.0.1, many more can be found in [2, §7.4]. Unlike ours, all of them are algebraic descriptions. The investigation of the consequences of this new characterization of Conradian orderings concerns almost two third of our work.

A major consequence of Theorem A is the classification of groups admitting only finitely many Conradian orderings. This can be thought of as an analogue of Tararin's classification of groups admitting only finitely many left-orderings [20, Theorem 5.2.1]. For the statement of both results, recall that a series

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

is said to be *rational* if it is subnormal (i.e., each  $G_i$  is normal in  $G_{i+1}$ ) and each quotient  $G_{i+1}/G_i$  is torsion-free rank-one Abelian. The series is called *normal* if, in addition, each  $G_i$  is normal in  $G$ . In §2.1 we show

**Theorem B.** *Let  $G$  be a  $\mathcal{C}$ -orderable group. If  $G$  admits only finitely many  $\mathcal{C}$ -orderings, then  $G$  admits a unique (hence normal) rational series. In this series, no quotient  $G_{i+2}/G_i$  is Abelian. Conversely, if  $G$  is a group admitting a normal rational series*

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

*so that no quotient  $G_{i+2}/G_i$  is Abelian, then the number of  $\mathcal{C}$ -orderings on  $G$  equals  $2^n$ .*

We state Tararin's classification as

**Theorem 1.1.1 (Tararin).** *Let  $G$  be a left-orderable group. If  $G$  admits only finitely many left-orderings, then  $G$  admits a unique (hence normal) rational series. In this series, no quotient  $G_{i+2}/G_i$  is bi-orderable. Conversely, if  $G$  is a group admitting a normal rational series*

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

*so that no quotient  $G_{i+2}/G_i$  is bi-orderable, then the number of left-orderings on  $G$  equals  $2^n$ .*

Note that the statement of Tararin's theorem is the same as that of Theorem B though changing " $\mathcal{C}$ -orderings" by "left-orderings", and the condition " $G_{i+2}/G_i$  non Abelian" by " $G_{i+2}/G_i$  non bi-orderable".

In the late nineties, using Tararin's classification, Zenkov was able to deduce that if a locally indicable group admits infinitely many left-orderings, then it admits uncountably many of them;

see [20, Theorem 5.2.5] or [41]. Here, in §2.2, we use our classification of groups admitting only finitely many  $\mathcal{C}$ -orderings to show

**Theorem C.** *Let  $G$  be a  $\mathcal{C}$ -orderable group. If  $G$  admits infinitely many  $\mathcal{C}$ -orderings, then it admits uncountably many  $\mathcal{C}$ -orderings. Moreover, none of these  $\mathcal{C}$ -orderings is isolated in the space of  $\mathcal{C}$ -orderings.*

We remark that the second statement of Theorem C is much stronger than the first one. For instance, if  $G$  is countable, then its space of  $\mathcal{C}$ -orderings is either finite or a Cantor set. Moreover, as it will be exemplified below, the absence of isolated  $\mathcal{C}$ -orderings when there are infinitely many of them, is a behavior not shared with left-orderings nor with bi-orderings. Actually, knowing when a given left-orderable group admits an isolated left-ordering is one of the main open problems in this theory.

Theorem C corroborates a general principle concerning  $\mathcal{C}$ -orderings. On the one hand, these are sufficiently rigid in that they allow deducing structure theorems for the underlying group (*e.g.*, local indicability). However, they are still sufficiently malleable in that, starting with a  $\mathcal{C}$ -ordering on a group, one may create very many  $\mathcal{C}$ -orderings, which turn out to be different from the original one with the only exception of the pathological cases described in Theorem B.

Motivated by the high regularity of Conradian orderings, in Chapter 3, we study the space of left-orderings of groups admitting only finitely many  $\mathcal{C}$ -orderings. This chapter is motivated by [36], where an explicit description of the space of left-orderings of the Baumslag-Solitar group  $B(1, 2)$  -a group with only  $2^2$   $\mathcal{C}$ -orderings, but infinitely many left-orderings- is made. In §3.1 we use the machinery developed/exposed in [29, §2], to extend the argument of [36], and give an explicit description of the space of left-orderings of any group admitting only four  $\mathcal{C}$ -orderings. We show that any group  $G$  admitting only four  $\mathcal{C}$ -orderings but infinitely left-orderings can be embedded in the (real) affine group, and that any left-ordering of  $G$  is an induced ordering (in the sense of §1.4) of this affine action, or one of the four possible Conradian orderings; see Theorem 3.1.4. Once the case “ $n = 2$ ” is solved, a simple induction argument shows

**Theorem D.** *If  $G$  is a  $\mathcal{C}$ -orderable group admitting only finitely many  $\mathcal{C}$ -orderings, then its space of left-orderings is either finite or homeomorphic to the Cantor set.*

As it was already mentioned, in the bi-ordered case the picture is totally different. At the time of this writing, there is no classification of groups admitting only finitely many bi-orderings. Actually the range of groups admitting only finitely many bi-orderings should be very large, and just a few results give partial descriptions of this situation; see for instance [2, Chapter VI] and [20, §5.3]. Indeed, this class contain all the groups fitting in Theorem B that are not in Tararin’s classification, but also a lot of groups algebraically very different from those. For example, the commutator subgroup of the group of piecewise affine homeomorphisms of the unit interval, and many other similar groups (such as the commutator subgroup of Thompson’s group  $F$ ), have finitely many bi-orderings; see [12, 42] and the remark at the end of Chapter 5. In addition, there are examples of bi-orderable groups admitting infinitely but (only) countably many bi-orderings; see [2, Chapter VI] or [5] for an example of a group admitting only countably infinitely many bi-orderings. For an example of a family of solvable groups admitting only finitely many bi-orderings see [20, §5.3]. This two “strange” behaviors are mainly caused by the strong rigidity of bi-invariant orderings, and our methods seem not well adapted to investigate this situation.

As mentioned earlier, for the case of left-orderings we have Tararin’s classification of groups admitting only finitely many left-orderings, and also Linnell’s result [23] saying that, if infinite, then the number of left-orderings admitted by a group must be uncountable. In §2.3, we give a new

proof of Linnell's result, but using a quite different approach. Our method relies strongly on the nature of Conradian orderings described in Theorem A. This new characterization of Conradian orderings is used to detect the so-called *Conradian soul of a left-ordering*  $\preceq$ , which was introduced by Navas in [31] as the maximal convex subgroup for which  $\preceq$  is Conradian; see §2.3.1. The Conradian soul plays a fundamental role when dealing with the problem of approximating a given left-ordering by its conjugates (in the sense of §1.3.1). We show that, in most cases, this can be done; see for instance Theorem 2.3.6. In the few cases this can not be done, we show that we still have enough information to conclude

**Theorem E (Linnell).** *If a left-orderable group admits infinitely many left-orderings, then it admits uncountably many left-orderings.*

To finish the discussion, we have to point out that the space of left-orderings of a group may be infinite and still have isolated left-orderings. Actually, this is the case of braid groups, [11, 13], and a particular central extension of Hecke groups [28].

In the last two chapters of our dissertation, we analyze the spaces of orderings of two remarkable groups.

Chapter 4 is devoted to the study of the space of left-orderings of the free group on two or more generators,  $F_n$ ,  $n \geq 2$ . This has a long history. In [24], McCleary studies an object introduced by Conrad in [9], called the free lattice-ordered group (in this case) over the free group, which is an universal object in the class of lattice-ordered groups<sup>3</sup>. He is able to prove that no left-ordering on the free group on two or more generators is *finitely determined*. In our language, this is equivalent to saying that the space of left-orderings of the free group on two or more generators has no isolated points. In [31], Navas gives a different and easier proof of this fact. He shows that small perturbations of the dynamical realization of a left-ordering of  $F_n$ , made outside large compact intervals in  $\mathbb{R}$ , can be used to approximate the given left-ordering.

In [8], Clay establishes a strong connection between some representations of the free lattice-ordered group over a group  $G$ , and the dynamics of the action of  $G$  on its space of left-orderings. Using this connection, together with the previous work of Kopitov [19], he showed

**Theorem F (Clay).** *The space of left-orderings of the free group on two or more generators has a dense orbit under the natural conjugacy action of  $F_n$ .*

However, Clay's proof is highly non constructive. Moreover, Kopytov's work [19] also involves the free lattice-ordered group over the free group. In Chapter 4 of this work, we give an explicit construction of a left-ordering of  $F_n$ , whose set of conjugates is dense in the space of left-orderings of  $F_n$ . Our proof uses a very simple idea which resembles a lot McCleary's and Kopitov's originals constructions from [24] and [19], respectively. Nevertheless, as we avoid the use of of any lattice structure, we don't have to take care of certain unpleasant technical details which make [24] and [19] hard to read. Thus, our construction is easier to follow.

The rough idea for proving Theorem F is the following. Since the space of left-orderings of  $F_n$  is compact, it contains a dense countable subset. Now, we can consider the dynamical realization of each of these left-orderings, and cut large pieces of each one of them. Since we are working with a free group, we can glue these pieces of dynamical realizations together in a sole action of  $F_n$  on the real line. Moreover, if the gluing is made with a little bit of care, then we can ensure very nice

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<sup>3</sup> A lattice-ordered group  $(G, \preceq)$ , is a group together with a partial bi-invariant ordering of  $G$  satisfying that for all  $f, g$  in  $G$  there exist  $f \vee g \in G$  (resp.  $f \wedge g \in G$ ), the least upper (resp. greatest lower) bound of  $f$  and  $g$ ; see for instance [15].

conjugacy properties, from which we can deduce Theorem F.

Theorem F gives a new proof of the fact that no left-ordering of  $F_n$  is isolated. Indeed, as also shown by Clay, any group  $G$  admitting a dense orbit in the natural action of  $G$  on its space of left-orderings has no isolated left-orderings. This is shown here in Proposition 4.0.4.

Finally, in Chapter 5, we study the space of bi-orderings of the (remarkable!) Thompson's group  $F$ . This group is usually represented as the group of piecewise affine homeomorphisms of the closed interval, such that the break points of any element in  $F$  are dyadic rationals, there are only finitely many of them, and the slopes of the elements are integer powers of 2. The standard reference on  $F$  is [6].

Using the theory of Conradian orderings together with the internal structure of Thompson's group  $F$ , we are able to show

**Theorem G.** *The space of bi-orderings of the Thompson group  $F$  is made up of eight isolated points, together with four canonical copies of the Cantor set.*

An important intermediate step in proving Theorem G, is the description of all bi-orderings on  $[F, F]$ , the commutator subgroup of  $F$ . We show that  $[F, F]$  admits only four bi-orderings. All of them can be easily described. This particular result is strongly related to [12], where Dlab shows that a large family of piecewise affine groups, where the slopes of the elements are contained in a rank-one Abelian group of the multiplicative groups of positive real numbers, have only four bi-orderings. However, the commutator subgroup of the Thompson group is not included in this family, since in [12], the “break points” of the elements considered can accumulate on the right. In particular, elements from [12] can have infinitely many break points. By counterpart, in Thompson's group  $F$ , the break points are contained in the dyadic numbers and, for each element, there are only finitely many of them. This additional assumption on the brake points implies for instance that conjugacy classes on  $F$  are way smaller than in the groups considered by Dlab. It also implies that  $F$  is not a lattice-ordered group under the natural pointwise partial ordering defined by  $f \succeq id$  if and only if  $f(x) \geq x$  for all  $x \in [0, 1]$ . In turns, the groups considered by Dlab are lattice-ordered with the point wise ordering; see [42] for a discussion on that subject (see also [25]). As a consequence, our method of proof is different from the one used in [12], and actually, the method for describing the bi-orderings on  $[F, F]$  is essentially the same as the one used to describe the bi-orderings on  $F$ .

We point out that in Chapter 5, besides the description of the topology of the space of bi-orderings of  $F$ , we give an explicit description of all the bi-orderings on  $F$ . The four canonical copies of the Cantor set arise from extending the bi-orderings on  $F/[F, F] \simeq \mathbb{Z}^2$  with each of the four bi-orderings on  $[F, F]$ . The eight isolated bi-orderings are explicitly described too. For instance, the left-ordering on  $F$ , whose set of positive elements are the elements whose first non trivial slope is greater than 1, determines a bi-invariant ordering, which is isolated in the space of bi-orderings. The other seven isolated bi-orderings are similar to this one, but we do not have space to describe them here.

We have compiled the main results of our work. The rest of Chapter 1 is devoted to give the basic background and notation. In §1.2 we illustrate some basic constructions for producing new orderings starting with a given one. In §1.3 we recall the concept of spaces of orderings. The main dynamical tools for carrying out our study are recalled in §1.4. More importantly, also in §1.4, we develop the concept of *crossings* for an action of a group on an ordered set, and we prove Theorem A.

## 1.2 Some basic constructions for producing new orderings

In this section we describe some basic constructions for creating new (left,  $\mathcal{C}$ , or bi) orderings starting with a given one. The main idea is to exploit the flexibility given by the convex subgroups.

If  $C$  is a proper convex subgroup of a left-ordered group  $(G, \preceq)$ , then  $\preceq$  induces a total order on the set of left-cosets of  $C$  by

$$g_1C \prec g_2C \Leftrightarrow g_1c_1 \prec g_2c_2 \text{ for all } c_1, c_2 \text{ in } C. \quad (1.1)$$

More importantly, this order is preserved by the left action of  $G$ ; see for instance [20, §2]. This easily implies

**Proposition 1.2.1.** *Let  $(G, \preceq)$  be a left-ordered (resp.  $\mathcal{C}$ -ordered) group and let  $C$  be a convex subgroup. Then any left-ordering (resp.  $\mathcal{C}$ -ordering) on  $C$  may be extended, via  $\preceq$ , to a total left-ordering (resp.  $\mathcal{C}$ -ordering) on  $G$ . In addition, in this new left-ordering,  $C$  is still a convex subgroup.*

*Proof:* We denote  $\preceq_1$  the induced ordering on the cosets of  $C$  (from equation (1.1)). Let  $\preceq_2$  be any left-ordering on  $C$ . For  $g \in G$ , we define  $id \preceq' g$  if and only if  $g \in C$  and  $id \preceq_2 g$  or  $g \notin C$  and  $C \prec_1 gC$ . We claim that  $\preceq'$  is a left-ordering.

Indeed, let  $f, h$  in  $G$  such that  $f \succ' id$  and  $h \succ' id$ . If both  $f, h$  belong to  $C$ , then clearly  $fh \in C$  and  $fh \succ' id$ . If neither  $f$  nor  $h$  belong to  $C$ , then, since the  $G$  action on the cosets of  $C$  preserves  $\preceq_1$ , we have that  $fhC \succ_1 C$ , thus  $fh \succ' id$ . Finally, if  $h \in C$  and  $f \notin C$ , we have that  $fC = fhC \succ_1 C$ , so  $fh \succ' id$ . To check that  $hf \succ' id$  we note that  $fC \succ_1 C$  implies  $hfC \succ_1 hC = C$ . This shows the left-invariance of  $\preceq'$ . To see that  $C$  is convex in  $\preceq'$ , we note that  $id \prec' h \prec' c$ , for  $c \in C$ , is equivalent to  $id \prec' h^{-1}c$ . We claim that in this case,  $h$  belongs to  $C$ . We have two possibilities. Either  $h^{-1}c \in C$ , in which case we conclude  $h \in C$ , or  $h^{-1}c \notin C$ , in which case we have that  $h^{-1}cC = h^{-1}C \succ_1 C$ , therefore,  $h^{-1} \succ' id$ , which contradicts the fact that  $id \prec' h$ . This shows the convexity of  $C$ .

We now show that  $\preceq'$  is Conradian when  $\preceq$  and  $\preceq_2$  are Conradian. Let  $f, g$  in  $G$ , be such that  $id \prec' f \prec' g$ . We have to show that  $fg^2 \succ' g$  (note that  $id \prec f \prec g$  easily implies  $gf^2 \succ f$  in any left-ordering!), or, equivalently, that  $g^{-1}fg^2 \succ' id$ . If both  $f$  and  $g$  belongs to  $C$ , then the conclusion follows by the assumption on  $\preceq_2$ . If it is the case that  $g$  does not belong to  $C$ , then we claim that  $g^{-1}fg^2 \notin C$ . Indeed, since  $\preceq$  is Conradian, the Conrad homomorphism  $\tau_{\preceq}^g$ , defined in Theorem 1.0.1, is an order preserving homomorphism whose kernel is  $G_g$ , and in this case  $C \subseteq G_g$ . Therefore  $\tau_{\preceq}^g(g^{-1}fg^2) = \tau_{\preceq}^g(fg) > 0$ . In particular  $g^{-1}fg^2 \notin C$ . This latter statement, together with the fact that  $\preceq$  is Conradian, implies  $g^{-1}fg^2C \succ_1 C$  which, in turns, implies  $g^{-1}fg^2 \succ' id$ .  $\square$

**Example 1.2.2.** Let  $\preceq$  be a left-ordering on  $G$ . Recall that the *reverse* (or “flipped”) ordering, denoted  $\overline{\preceq}$ , is the ordering that satisfies  $f \overline{\preceq} g \Leftrightarrow f \succ g$ . Showing that if  $\preceq$  is a left,  $\mathcal{C}$ , or bi- ordering then  $\overline{\preceq}$  is of the same kind is routine. Moreover, the convex series in  $\preceq$  coincides with the convex series in  $\overline{\preceq}$ . Now, suppose there is a nontrivial convex subgroup  $C$  of  $G$ . Then, by Proposition 1.2.1, there is a (left- or  $\mathcal{C}$ -) ordering  $\preceq_C$  of  $G$  defined by  $id \prec_C f$ , where  $f \in G$ , if and only if either

- $f \succ id$  and  $f \notin C$ , or
- $f \overline{\preceq} id$  and  $f \in C$ .

In the case  $\preceq$  is a bi-ordering, the preceding construction does not imply that  $\preceq_C$  is also a bi-ordering. Nevertheless, if  $C$  is a convex and normal subgroup, then the conclusion follows. Indeed, for  $f, g$  in  $G$  with  $id \prec_C f$ , we have to prove that  $id \prec_C gfg^{-1}$ . If  $f \notin C$ , then by the normality of

$C$  we have that  $gfg^{-1} \notin C$  and the conclusion follows from the bi-invariance of  $\preceq$ . If  $f \in C$ , then we have that  $id \succ f$  and  $id \succ gfg^{-1} \in C$ , which is the same to say that  $id \prec_C gfg^{-1}$ , so  $\preceq_C$  is a bi-ordering.

The following example will serve us to approximate (in the sense of §1.3) a given ordering when the series of convex subgroups is long enough.

**Example 1.2.3.** Let  $g \in G \setminus \{id\}$  and let  $\preceq$  be a left-ordering on  $G$ . Consider the (perhaps infinite) series of  $\preceq$ -convex subgroups.

$$\{id\} = G^{id} \subset \dots \subset G_g \subset G^g \subset \dots \subset G.$$

We will use Example 1.2.2 to produce the a new ordering  $\preceq^g$  by “flipping” the ordering on  $G^g \setminus G_g$ . More precisely, we let  $\preceq^g = (\preceq_{G^g})_{G_g}$ , that is, we flip the ordering on  $G^g$  and then we flip again on  $G_g$ . It is easy to see that, for  $f \in G$ ,  $id \prec^g f$  if and only if

- $f \succ id$  and  $f \notin G^g$ ,
- $f \succ id$  and  $f \in G^g \setminus G_g$ ,
- $f \succ id$  and  $f \notin G_g$ .

Clearly,  $\preceq^g$  is Conradian when  $\preceq$  is Conradian.

Notice that if  $C$  is normal in  $G$ , equation (1.1) defines a left-ordering on the group  $G/C$ . In this case, we have even more flexibility for producing new (left-,  $\mathcal{C}$ - or bi-) orderings, since we can change our ordering not only on the subgroup  $C$ , but also on the quotient group  $G/C$ . We state this as

**Lemma 1.2.4.** Let  $(G, \preceq)$  be a left-ordered group. Let  $C$  be a normal and convex subgroup of  $G$ . We denote by  $\preceq_1$  the induced ordering on  $G/C$  (from equation (1.1)) and by  $\preceq_2$  the restriction to  $C$  of  $\preceq$ . We have,

- (i) For  $f \in G$ ,  $id \prec f$  if and only if  $f \notin C$  and  $fC \succ_1 C$ , or  $f \in C$  and  $f \succ_2 id$ .
- (ii) Let  $f \in G$ . For any left-ordering  $\preceq''$  on  $C$ , and any left-ordering  $\preceq'$  on  $G/C$ , there is a left-ordering  $\tilde{\preceq}$  on  $G$  defined by  $f \tilde{\succ} id$  if and only if  $f \notin C$  and  $fC \succ' C$ , or  $f \in C$  and  $f \succ'' id$ .
- (iii)  $\preceq$  is Conradian if and only if  $\preceq_1$  and  $\preceq_2$  are Conradian.
- (iv)  $\preceq$  is a bi-ordering if and only if  $\preceq_1$  and  $\preceq_2$  are bi-orderings and for  $c \in C$ ,  $id \prec_2 c \Rightarrow id \prec_2 fcf^{-1}$  for all  $f \in G$ .

*Proof:* Items (i), (ii) and (iii) follow arguing as in the proof of Proposition 1.2.1.

We show item (iv). Clearly, if  $\preceq$  is a bi-ordering, then  $\preceq_2$  is invariant under the whole group  $G$ . That is,  $c \in C$  and  $id \prec_2 c$  implies  $fcf^{-1} \in C$  and  $id \prec_2 fcf^{-1}$  for all  $f \in G$ . To see that  $\preceq_1$  is bi-invariant we note that, if not, then there are  $f, g$  in  $G$  such that  $C \preceq_1 fC$  and  $gCfCg^{-1}C = gfg^{-1}C \prec_1 C$ . In particular,  $gfg^{-1} \notin C$ , and item (i) implies  $gfg^{-1} \prec id$ . This contradiction implies that  $\preceq_1$  is a bi-ordering.

For the converse, suppose that  $\preceq$  is not a bi-ordering. Then there are  $f$  and  $g$  in  $G$  such that  $id \prec f$  and  $gfg^{-1} \prec id$ . If  $f \in C$  we have that  $\preceq_2$  is not invariant under the action of  $g$ , which contradicts the assumption on  $\preceq_2$ . If  $f \notin C$  then, since  $C$  is a normal subgroup,  $gfg^{-1} \notin C$ . But, by item (i), in this case we have that  $C \prec_1 fC$  and  $gfg^{-1}C \prec_1 C$ , so  $\preceq_1$  is not bi-invariant, contrary to our assumption on  $\preceq_2$ .  $\square$

What follows is a rewording of the previous lemma.

**Corollary 1.2.5.** *Suppose that  $G$  is a left-orderable (resp.  $\mathcal{C}$ -orderable) group and  $C$  is a normal subgroup of  $G$ . Then for any left-ordering (resp.  $\mathcal{C}$ -ordering)  $\preceq_1$  on  $G/C$  and any left-ordering (resp.  $\mathcal{C}$ -ordering)  $\preceq_2$  on  $C$ , there is a left-ordering (resp.  $\mathcal{C}$ -ordering)  $\preceq$  on  $G$  such that  $\preceq$  coincides with  $\preceq_2$  on  $C$  and the induced ordering of  $\preceq$  on  $G/C$  coincides with  $\preceq_1$ .*

*If, in addition,  $G$  is bi-orderable, then for any bi-ordering  $\preceq_1$  of  $G/C$  and any bi-ordering  $\preceq_2$  of  $C$  with the additional property that, for  $c \in C$  and  $f \in G$  we have  $id \prec_2 c \Rightarrow id \prec_2 fcf^{-1}$ , there is a bi-ordering  $\preceq$  on  $G$  such that  $\preceq$  coincides with  $\preceq_2$  on  $C$  and the induced ordering of  $\preceq$  on  $G/C$  coincides with  $\preceq_1$ .*

### 1.3 The space of orderings of a group

Recall that, given a left-ordering  $\preceq$  on a group  $G$ , we say that  $f \in G$  is *positive* or  $\preceq$ -*positive* (resp. *negative* or  $\preceq$ -*negative*) if  $f \succ id$  (resp.  $f \prec id$ ). We denote  $P_{\preceq}$  the set of  $\preceq$ -positive elements in  $G$ , and we sometimes call it *the positive cone* of  $\preceq$ . Clearly,  $P_{\preceq}$  satisfies the following properties:

- (i)  $P_{\preceq}P_{\preceq} \subseteq P_{\preceq}$ , that is,  $P_{\preceq}$  is a semi-group;
- (ii)  $G = P_{\preceq} \sqcup P_{\preceq}^{-1} \sqcup \{id\}$ , where the union is disjoint, and  $P_{\preceq}^{-1} = \{g^{-1} \in G \mid g \in P_{\preceq}\} = \{g \in G \mid g \prec id\}$ .

Moreover, given any subset  $P \subseteq G$  satisfying the conditions (i) and (ii) above, we can define a left-ordering  $\preceq_P$  by  $f \prec_P g$  if and only if  $f^{-1}g \in P$ . We will usually identify  $\preceq$  with its positive cone  $P_{\preceq}$ .

Given a left-orderable group  $G$  (of arbitrary cardinality), we denote the set of all left-orderings on  $G$  by  $\mathcal{LO}(G)$ . This set has a natural topology first introduced by Sikora in [37]. This topology can be defined by identifying  $P \in \mathcal{LO}(G)$  with its characteristic function  $\chi_P \in \{0, 1\}^G$ . In this way, we can view  $\mathcal{LO}(G)$  embedded in  $\{0, 1\}^G$ . This latter space, with the product topology, is a Hausdorff, totally disconnected, and compact space. It is not hard to see that (the image of)  $\mathcal{LO}(G)$  is closed inside, and hence compact as well (see [31, 37] for details).

A basis of neighborhoods of  $\preceq$  in  $\mathcal{LO}(G)$  is the family of the sets  $U_{g_1, \dots, g_k}$  of all left-orderings  $\preceq'$  on  $G$  that coincide with  $\preceq$  on  $\{g_1, \dots, g_k\}$ , where  $\{g_1, \dots, g_k\}$  runs over all finite subsets of  $G$ . Another basis of neighborhoods is given by the sets  $V_{f_1, \dots, f_k}$  of all left-orderings  $\preceq'$  on  $G$  such that all the  $f_i$  are  $\preceq'$ -positive, where  $\{f_1, \dots, f_k\}$  runs over all finite subsets of  $\preceq$ -positive elements of  $G$ . The (perhaps empty) subspaces  $\mathcal{BO}(G)$  and  $\mathcal{CO}(G)$  of bi-orderings and  $\mathcal{C}$ -orderings on  $G$  respectively, are closed inside  $\mathcal{LO}(G)$ , hence compact; see [31].

If  $G$  is countable, then this topology is metrizable: given an exhaustion  $G_0 \subset G_1 \subset \dots$  of  $G$  by finite sets, for different  $\preceq$  and  $\preceq'$ , we may define  $dist(\preceq, \preceq') = 1/2^n$ , where  $n$  is the first integer such that  $\preceq$  and  $\preceq'$  do not coincide on  $G_n$ . If  $G$  is finitely generated, we may take  $G_n$  as the ball of radius  $n$  with respect to a fixed finite system of generators.

**Example 1.3.1.** It was shown in [37] that the space of (bi-) orderings on a torsion free Abelian group of rank greater than one is homeomorphic to the Cantor set. We now describe the space of orderings of  $\mathbb{Z}^2$ .

Let  $e_1, e_2$  be the standard basis on  $\mathbb{Z}^2$ . For every  $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  define  $\psi_x : \mathbb{Z}^2 \rightarrow \mathbb{R}$  to be the homomorphism defined by  $\psi_{\infty}(e_1) = 0$ ,  $\psi_{\infty}(e_2) = 1$  and  $\psi_x(e_1) = 1$ ,  $\psi_x(e_2) = x$  if  $x \in \mathbb{R}$ .

Clearly, if  $x$  is irrational, then  $\psi_x$  is injective and  $P_x = \{g \in \mathbb{Z}^2 \mid \psi_x(g) > 0\}$  defines a positive cone in  $\mathbb{Z}^2$ . The associated ordering is said to be of *irrational type*. Note that this ordering has no proper convex subgroup. These irrational orderings are dense in  $\mathcal{LO}(\mathbb{Z}^2)$ .

If  $x$  is rational or  $x = \infty$ , then  $\psi_x$  is not injective and  $\ker(\psi_x) \simeq \mathbb{Z}$ . Thus, the set  $P_x = \{g \in \mathbb{Z}^2 \mid \psi_x(g) > 0\}$  defines only a partial ordering which can be completed into two total orderings  $P_x^+$  and  $P_x^-$ , where  $P_x^+$  (resp.  $P_x^-$ ) corresponds to the limit of  $P_{x_n}$  where  $(x_n)$  is a sequence of irrational numbers converging to  $x$  from the right (resp. left). These orderings are called of *rational type* (e.g., a lexicographic ordering). In these orderings,  $\ker(\psi_x)$  is the unique proper convex subgroup.

Finally, we show that any ordering on  $\mathbb{Z}^2$  is one of the orderings just described. Let  $\preceq$  be any ordering on  $\mathbb{Z}^2$ . Since  $\mathbb{Z}^2$  is finitely generated and Abelian,  $\preceq$  is Conradian, and, for  $g = \max_{\preceq} \{\pm e_1, \pm e_2\}$ , there exists  $\tau = \tau_{\preceq}^g : \mathbb{Z}^2 \rightarrow \mathbb{R}$  (as defined in Theorem 1.0.1). Now, let

$$y = \begin{cases} \tau(e_2)/\tau(e_1) & \text{if } \tau(e_1) \neq 0, \\ \infty & \text{if } \tau(e_1) = 0. \end{cases}$$

Then there is a positive real number  $\alpha$  such that  $\alpha\tau = \psi_y$ . This shows that  $\preceq$  must coincide with  $P_y$  or  $P_y^\pm$ .

There is another, more geometric, way to see the orderings on  $\mathbb{Z}^2$ . Since the orderings of rational type are limits of orderings of irrational type, we just describe the latter type of orderings. Let  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Consider  $\mathbb{Z}^2$  embedded in  $\mathbb{R}^2$  in the usual way. The unique  $\mathbb{R}$ -linear extension of  $\psi_x$  from  $\mathbb{Z}^2$  to  $\mathbb{R}^2$  will be denoted  $\hat{\psi}_x$ . Let  $L_x = \{w \in \mathbb{R}^2 \mid \hat{\psi}_x(w) = 0\}$ . Take  $w_0 \in \hat{\psi}_x^{-1}(1)$  and let  $H_x$  be the open half plane with boundary  $L_x$  that contains  $w_0$ . Then we have that  $P_x = \mathbb{Z}^2 \cap H_x$ . Moreover, if  $\preceq$  is the ordering on  $\mathbb{Z}^2$  corresponding to  $P_x$ , then for  $g_1$  and  $g_2$  in  $P_x$ , we have that  $g_1 \prec g_2 \Leftrightarrow \text{dist}(g_1, L_x) < \text{dist}(g_2, L_x)$ , where  $\text{dist}$  is the Euclidean distance in  $\mathbb{R}^2$ .

### 1.3.1 An action on the space of orderings

One of the most interesting properties of  $\mathcal{LO}(G)$  is that  $\text{Aut}(G)$ , the group of automorphism of  $G$ , naturally acts on  $\mathcal{LO}(G)$  by continuous transformations. More precisely, given any  $\varphi \in \text{Aut}(G)$ , we define  $\varphi(\preceq) = \preceq_\varphi \in \mathcal{LO}(G)$  by letting  $h \preceq_\varphi f$  if and only if  $\varphi^{-1}(h) \prec \varphi^{-1}(f)$ , where  $h, f$  belong to  $G$ . One easily checks that  $\varphi(U_{g_1, \dots, g_n}) = U_{\varphi(g_1), \dots, \varphi(g_n)}$ .

In particular, we obtain an action of  $G$  on  $\mathcal{LO}(G)$  which factors throughout the group of *inner automorphisms*<sup>4</sup>. The above condition reads  $g(\preceq) = \preceq_g$ , where by definition,  $h \succ_g f$  if and only if  $ghg^{-1} \succ gfg^{-1}$ . We say that  $\preceq_g$  is the conjugate of  $\preceq$  under  $g$ .

It immediately follows that the global fixed points of  $G$  for this action are precisely the bi-orderings of  $G$ . In particular, the action of  $\text{Aut}(G)$  on  $\mathcal{BO}(G)$  factors throughout  $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ , the group of outer automorphism of  $G$ .

Another typical object in dynamics, namely periodic orbits, also plays an important role in the theory of orderable groups.

**Proposition 1.3.2.** *In the action of  $G$  on  $\mathcal{LO}(G)$ , every periodic point (that is, a point whose orbit is finite) corresponds to a  $\mathcal{C}$ -ordering.*

*Proof:* Suppose that  $\preceq$  has a periodic orbit. Then  $\text{Stab}_G(\preceq) = \{g \in G \mid g(\preceq) = \preceq\}$  has finite index in  $G$ . Moreover, the restriction of  $\preceq$  to  $\text{Stab}_G(\preceq)$  is a bi-invariant ordering. In particular it is Conradian. Now, by a result of Rhemtulla and Rolfsen [35, Theorem 2.4], here Corollary 2.1.2, we have that  $\preceq$  is Conradian.  $\square$

One may wish that any Conradian ordering is periodic, but, as shown in [27, Example 4.6], there is a group such that no Conradian ordering is periodic. Our next example shows that on the Heisenberg group -a group in which every left-ordering is Conradian- both phenomena appears, i.e. it admits periodic and non periodic orbits.

<sup>4</sup>Inner automorphisms are automorphisms induced by conjugation of  $G$ .



**Example 1.3.3.** Let  $H = \langle a, b, c \mid [a, b] = c, ac = ca, bc = cb \rangle$  be the discrete Heisenberg group.

The group  $H$  is left-orderable, since  $H/[H, H] \simeq \mathbb{Z}^2$  and  $[H, H] \simeq \mathbb{Z}$ . Therefore, we can produce an ordering  $\preceq$  on  $H$  by defining an ordering  $\preceq_1$  on  $H/[H, H]$  and an ordering  $\preceq_2$  on  $[H, H]$ , and then declaring  $id \prec g$  if and only if  $[H, H] \prec_1 g[H, H]$  or  $g \in [H, H]$  and  $id \prec_2 g$ . The ordering  $\preceq$  is easily shown to be bi-invariant, so it is a fixed point for the action of  $G$  in  $\mathcal{LO}(G)$ . Moreover, due to Proposition 2.2.2, we have that every ordering on  $H$  is Conradian.

Note that we have the freedom to choose any ordering on  $H/[H, H] \simeq \mathbb{Z}^2$ , so we can assume that  $\langle b[H, H] \rangle \triangleleft H/[H, H]$  is convex in  $\preceq_1$ . This implies that in the ordering  $\preceq$ , the subgroup  $\langle b, c \rangle \simeq \mathbb{Z}^2$  is normal and convex in  $H$  (and  $[H, H] = \langle c \rangle$  is convex in  $\langle b, c \rangle$ ). Then, using Corollary 1.2.5, we can define an ordering  $\preceq'$  on  $H$  by choosing any ordering  $\preceq'_2$  on  $\langle b, c \rangle \simeq \mathbb{Z}^2$  and an ordering  $\preceq'_1$  on  $H/\langle b, c \rangle \simeq \mathbb{Z}$ . For concreteness, we let  $\preceq'_2$  be an ordering of irrational type of  $\mathbb{Z}^2$ , say  $P_x$ ,  $x \in \mathbb{R} \setminus \mathbb{Q}$ ; see Example 1.3.1.

We now let  $X \subset \mathcal{LO}(H)$  be the orbit of  $\preceq'$ . Since  $\langle b, c \rangle$  is normal, convex and Abelian, it acts trivially on  $X$ . Therefore,  $X = \{\preceq'_{a^n} \mid n \in \mathbb{Z}\}$ . To see that  $\preceq'$  is not periodic, it is enough to see that, if  $n \neq 0$ , then the restrictions of  $\preceq'_{a^n}$  and  $\preceq'$  to  $\langle b, c \rangle$  do not coincide. To see this, note that, since  $aba^{-1} = bc$ , and  $aca^{-1} = c$ , making the identifications  $e_1 = c$  and  $e_2 = b$ , we have that the action by conjugation of  $a$  on  $\langle b, c \rangle$  corresponds to the action on  $\mathbb{Z}^2 \subset \mathbb{R}^2$  given by the matrix

$$M_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now, recall from Example 1.3.1 that to any ordering on  $\mathbb{Z}^2$  we associate a one dimensional vector space  $L_x \subset \mathbb{R}^2$ , namely the kernel of the  $\mathbb{R}$ -linear map  $e_1 \mapsto 1$ ,  $e_2 \mapsto x$ . Moreover, if  $L_x \neq L_y$  then  $P_x \neq P_y$ . Checking that  $M_a^n(L_x) \neq L_x$  for  $x \in \mathbb{R} \setminus \mathbb{Q}$  and every  $n \neq 0$  is an easy exercise. In particular,  $\preceq'$  is not periodic.

## 1.4 Dynamical versions of group orderability

Though orderability may look as a very algebraic concept, it has a deep (one-dimensional) dynamical content. The following theorem, due to P. Cohn, M. Zaitseva, and P. Conrad, goes in this direction (see [20, Theorem 3.4.1]):

**Theorem 1.4.1.** *A group  $G$  is left-orderable if and only if it embeds in the group of (order-preserving) automorphisms of a totally ordered set.*

Both implications of this theorem are easy. In one direction, note that a left-ordered group acts on itself by order preserving automorphisms, namely, left translations. Conversely, to create a left-ordering on a group  $G$  of automorphisms of a totally ordered set  $(\Omega, \leq)$ , we construct the what is called *induced ordering* from the action as follows. Fix a well-order  $\leq^*$  on the elements of  $\Omega$ , and, for every  $f \in G$ , let  $w_f = \min_{\leq^*} \{w \in \Omega \mid f(w) \neq w\}$ . Then we define an ordering  $\preceq$  on  $G$  by letting  $f \succ id$  if and only if  $f(w_f) > w_f$ . It is not hard to check that this order relation is a (total) left-ordering on  $G$ .

For the case of countable groups, we can give more dynamical information since we can take  $\Omega$  as being the real line (see [14, Theorem 6.8], [29, §2.2.3], or [31] for further details). This characterization will be used in §3 and §4.1.

**Proposition 1.4.2.** *For a countable infinite group  $G$ , the following two properties are equivalent:*

- $G$  is left-orderable,
- $G$  acts faithfully on the real line by orientation-preserving homeomorphisms. That is, there is an homomorphic embedding  $G \rightarrow \text{Homeo}_+(\mathbb{R})$ .

*Sketch of proof:* The fact that a group of orientation-preserving homeomorphisms of the real line is left-orderable is a direct consequence of Theorem 1.4.1.

For the converse, we construct what is called a *dynamical realization of a left-ordering*  $\preceq$ . Fix an enumeration  $(g_i)_{i \geq 0}$  of  $G$ , and let  $t_{\preceq}(g_0) = 0$ . We shall define an order-preserving map  $t_{\preceq} : G \rightarrow \mathbb{R}$  by induction. Suppose that  $t_{\preceq}(g_0), t_{\preceq}(g_1), \dots, t_{\preceq}(g_i)$  have been already defined. Then if  $g_{i+1}$  is greater (resp. smaller) than all  $g_0, \dots, g_i$ , we define  $t_{\preceq}(g_{i+1}) = \max\{t_{\preceq}(g_0), \dots, t_{\preceq}(g_i)\} + 1$  (resp.  $\min\{t_{\preceq}(g_0), \dots, t_{\preceq}(g_i)\} - 1$ ). If  $g_{i+1}$  is neither greater nor smaller than all  $g_0, \dots, g_i$ , then there are  $g_n, g_m \in \{g_0, \dots, g_i\}$  such that  $g_n \prec g_{i+1} \prec g_m$  and no  $g_j$  is between  $g_n, g_m$  for  $0 \leq j \leq i$ . Then we set  $t_{\preceq}(g_{i+1}) = (t_{\preceq}(g_n) + t_{\preceq}(g_m))/2$ .

Note that  $G$  acts naturally on  $t_{\preceq}(G)$  by  $g(t_{\preceq}(g_i)) = t_{\preceq}(gg_i)$ . It is not difficult to see that this action extends continuously to the closure of  $t_{\preceq}(G)$ . Finally, one can extend the action to the whole real line by declaring the map  $g$  to be affine on each interval in the complement of  $\overline{t_{\preceq}(G)}$ .  $\square$

We have constructed an embedding  $G \rightarrow \text{Homeo}_+(\mathbb{R})$ . We call this embedding a dynamical realization of the left-ordered group  $(G, \preceq)$ . The order preserving map  $t_{\preceq}$  is called the reference map. When the context is clear, we will drop the subscript  $\preceq$  of the map  $t_{\preceq}$ .

**Remark 1.4.3.** As constructed above, the dynamical realization depends not only on the left-ordering  $\preceq$ , but also on the enumeration  $(g_i)_{i \geq 0}$ . Nevertheless, it is not hard to check that dynamical realizations associated to different enumerations (but the same ordering) are *topologically conjugate*.<sup>5</sup> Thus, up to topological conjugacy, the dynamical realization depends only on the ordering  $\preceq$  of  $G$ .

An important property of dynamical realizations is that they do not admit global fixed points (i.e., no point is stabilized by the whole group). Another important property is that  $t_{\preceq}(id)$  has a *free orbit* (i.e.  $\{g \in G \mid g(t_{\preceq}(id)) = t_{\preceq}(id)\} = \{id\}$ ). Hence  $g \succ id$  if and only if  $g(t_{\preceq}(id)) > t_{\preceq}(id)$ , which allows us to recover the left-ordering from its dynamical realization.

### 1.4.1 Crossings: a new characterization of Conrad's property

The Conrad property has many characterizations; see Theorem 1.0.1 and [2, §7.4]. All of them are algebraic descriptions. We finish this introductory chapter giving a new characterization of the Conrad property for left-orderings which has a strong dynamical flavor. The dynamical object to look at are the so-called *crossings*. We will make a strong use of this characterization in Chapter 2.

The following definition, first introduced in [31] for the case of countable groups and latter in [33] for the general case, will be of great importance in this work. Let  $G$  be a group acting by order preserving bijections on a totally ordered space  $(\Omega, \leq)$ . A *crossing* for the action of  $G$  on  $\Omega$  is a 5-uple  $(f, g, u, v, w)$  where  $f, g$  (resp.  $u, v, w$ ) belong to  $G$  (resp.  $\Omega$ ) and satisfy:

- (i)  $u < w < v$ .
- (ii) For every  $n \in \mathbb{N}$ , we have  $g^n u < v$  and  $f^n v > u$ .
- (iii) There exist  $M, N$  in  $\mathbb{N}$  such that  $f^N v < w < g^M u$ .

---

<sup>5</sup>Two actions  $\phi_1 : G \rightarrow \text{Homeo}_+(\mathbb{R})$  and  $\phi_2 : G \rightarrow \text{Homeo}_+(\mathbb{R})$  are topologically conjugate if there exists  $\varphi \in \text{Homeo}_+(\mathbb{R})$  such that  $\varphi \circ \phi_1(g) = \phi_2(g) \circ \varphi$  for all  $g \in G$ .

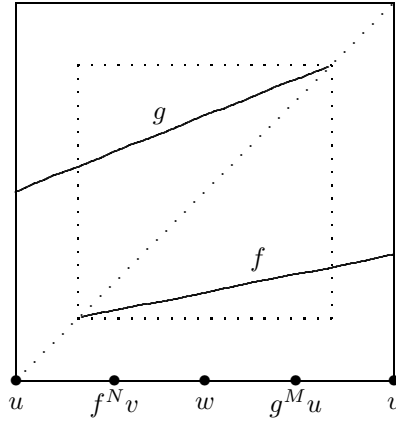


Figure 1.1: A crossing

The reason why this definition is so important is because it actually characterizes the  $\mathcal{C}$ -orderings, as is shown in [33, Theorem 1.4]. We quote the theorem and the proof below.

**Theorem A.** *A left-ordering  $\preceq$  on  $G$  is Conradian if and only if the action of  $G$  by left translations on itself admits no crossing.*

*Proof:* Suppose that  $\preceq$  is not Conradian, and let  $f, g$  be positive elements so that  $fg^n \prec g$  for every  $n \in \mathbb{N}$ . We claim that  $(f, g, u, v, w)$  is a crossing for  $(G, \preceq)$  for the choice  $u = 1$ , and  $v = f^{-1}g$ , and  $w = g^2$ . Indeed:

- From  $fg^2 \prec g$  one obtains  $g^2 \prec f^{-1}g$ , and since  $g \succ 1$ , this gives  $1 \prec g^2 \prec f^{-1}g$ , that is,  $u \prec w \prec v$ .
- From  $fg^n \prec g$  one gets  $g^n \prec f^{-1}g$ , that is,  $g^nu \prec v$  (for every  $n \in \mathbb{N}$ ); on the other hand, since both  $f, g$  are positive, we have  $f^{n-1}g \succ 1$ , and thus  $f^n(f^{-1}g) \succ 1$ , that is,  $f^nv \succ u$  (for every  $n \in \mathbb{N}$ ).
- The relation  $f(f^{-1}g) = g \prec g^2$  may be read as  $f^N v \prec w$  for  $N = 1$ ; on the other hand, the relation  $g^2 \prec g^3$  is  $w \prec g^M u$  for  $M = 3$ .

Conversely, assume that  $(f, g, u, v, w)$  is a crossing for  $(G, \preceq)$  so that  $f^N v \prec w \prec g^M u$  (with  $M, N$  in  $\mathbb{N}$ ). We will prove that  $\preceq$  is not Conradian by showing that, for  $h = g^M f^N$  and  $\bar{h} = g^M$ , both elements  $w^{-1}hw$  and  $w^{-1}\bar{h}w$  are positive, but

$$(w^{-1}hw)(w^{-1}\bar{h}w)^n \prec w^{-1}\bar{h}w \quad \text{for all } n \in \mathbb{N}.$$

To show this, first note that  $gw \succ w$ . Indeed, if not, then we would have

$$w \prec g^N u \prec g^N w \prec g^{N-1}w \prec \dots \prec gw \prec w,$$

which is absurd. Clearly, the inequality  $gw \succ w$  implies

$$g^M w \succ g^{M-1}w \succ \dots \succ gw \succ w,$$

and hence

$$w^{-1}\bar{h}w = w^{-1}g^M w \succ 1. \tag{1.2}$$

Moreover,

$$hw = g^M f^N w \succ g^M f^N f^N v = g^M f^{2N} v \succ g^M u \succ w.$$

and hence

$$w^{-1}hw \succ 1. \quad (1.3)$$

Now note that for every  $n \in \mathbb{N}$ ,

$$h\bar{h}^nw = hg^{Mn}w \prec hg^{Mn}g^Mu = hg^{Mn+M}u \prec hv = g^Mf^Nv \prec g^Mw = \bar{h}w.$$

After multiplying by the left by  $w^{-1}$ , the last inequality becomes

$$(w^{-1}hw)(w^{-1}\bar{h}w)^n = w^{-1}h\bar{h}^nw \prec w^{-1}\bar{h}w,$$

as we wanted to check. Together with (1.2) and (1.3), this shows that  $\preceq$  is not Conradian.  $\square$

## Chapter 2

# Applications of the new characterization of Conrad's property

With the aid of Theorem A, we are able to prove two structure theorems for  $\mathcal{C}$ -orderings. One is the algebraic description of groups with finitely many  $\mathcal{C}$ -orderings; Theorem B in §2.1. The second tell us that the spaces of Conradian groups orderings are either finite or without isolated points; Theorem C in §2.2. This is essentially taken from [36]. As an application, we give a new proof of a theorem of Navas in [31] asserting that torsion free nilpotent groups have no isolated left-orderings unless they are rank-one Abelian; see §2.2.2.

Theorem A also serves us to detect the so-called *Conradian soul of a left-ordering*; see §2.3.1. This allows us to ensure many good conjugacy properties of the left action of the group on itself and to investigate the possibility of approximating a given left-ordering by its conjugates, which in many cases can be done; §2.3.2. This is used to give a new proof of the fact, first proved by Linnell, that a left-orderable group admits either finitely or uncountably many left-orderings; Theorem E in §2.3.3. This is essentially taken from [33], which, in turn, is inspired from [31].

### 2.1 On groups with finitely many Conradian orderings

In this section we give an algebraic description of groups admitting only finitely many Conradian orderings [36]. This may be considered as an analogue of Tararin's classification of groups admitting only finitely many left-orderings; see Theorem 1.1.1 or [20, Theorem 5.2.1]. For the statement, recall that a series

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

is said to be *rational* if it is subnormal (*i.e.*, each  $G_i$  is normal in  $G_{i+1}$ ) and each quotient  $G_{i+1}/G_i$  is torsion-free rank-one Abelian. The series is called *normal* if, in addition, each  $G_i$  is normal in  $G$ .

**Theorem B.** *Let  $G$  be a  $\mathcal{C}$ -orderable group. If  $G$  admits finitely many  $\mathcal{C}$ -orderings, then  $G$  admits a unique (hence normal) rational series. In this series, no quotient  $G_{i+2}/G_i$  is Abelian. Conversely, if  $G$  is a group admitting a normal rational series*

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

*so that no quotient  $G_{i+2}/G_i$  is Abelian, then the number of  $\mathcal{C}$ -orderings on  $G$  equals  $2^n$ .*

### 2.1.1 Some lemmata

The following crucial lemma is essentially proved by Navas in [31] in the case of countable groups, but the proof therein rests upon very specific issues about the dynamical realization of an ordered group. Here we give a general algebraic proof.

**Lemma 2.1.1.** *Suppose  $G$  is faithfully acting by order preserving bijections on a totally ordered set  $(\Omega, \leq)$ . Then, the action has no crossings if and only if any induced ordering is Conradian.*

*Proof:* Suppose that the ordering  $\preceq$  on  $G$  induced from some well-order  $\leq^*$  on  $\Omega$  is not Conradian (comments after Theorem 1.4.1 explain how to induce an ordering from an action, and shows the relation between  $\preceq$  and  $\leq^*$ ). Then there are  $\preceq$ -positive elements  $f, g$  in  $G$  such that  $fg^n \prec g$ , for every  $n \in \mathbb{N}$ . This easily implies  $f \prec g$ , since, in the case  $id \prec g \prec f$ , we get  $id \prec g \prec g^n$ , so  $id \prec g \prec f \prec fg \prec fg^n$ .

Let  $\bar{w} = \min_{\leq^*} \{w_f, w_g\}$ , where, for  $h \in G$ ,  $w_h = \min_{\leq^*} \{w \in \Omega \mid h(w) \neq w\}$ . We claim that  $(fg, fg^2, \bar{w}, g(\bar{w}), fg^2(\bar{w}))$  is a crossing (see Figure 2.1). Indeed, the inequalities  $id \prec f \prec g$  imply that  $\bar{w} = w_g \leq^* w_f$  and  $g(\bar{w}) > \bar{w}$ . Moreover,  $f(\bar{w}) \geq \bar{w}$ , which together with  $fg^n \prec g$  yield  $\bar{w} < fg^2(\bar{w}) < g(\bar{w})$ , hence condition (i) of the definition of crossing is satisfied. Note that the preceding argument actually shows that  $fg^n(\bar{w}) < g(\bar{w})$ , for all  $n \in \mathbb{N}$ . Thus  $fg^2fg^2(\bar{w}) < fg^3(\bar{w}) < g(\bar{w})$ . A straightforward induction argument shows that  $(fg^2)^n(\bar{w}) < g(\bar{w})$ , for all  $n \in \mathbb{N}$ , which proves the first part of condition (ii). For the second part, from  $g(\bar{w}) > \bar{w}$  and  $f(\bar{w}) \geq \bar{w}$  we conclude that  $\bar{w} < (fg)^n(g(\bar{w}))$ . Condition (iii) follows because  $\bar{w} < fg^2(\bar{w})$  implies  $fg^2(\bar{w}) < fg^2(fg^2(\bar{w})) = (fg^2)^2(\bar{w})$ , and  $fg^2(\bar{w}) < g(\bar{w})$  implies  $(fg^2)^2(g(\bar{w})) = fg(fg^2(\bar{w})) < fg(g(\bar{w})) = fg^2(\bar{w})$ .

For the converse, suppose that  $(f, g, u, v, w)$  is a crossing for the action. Let  $N, M$  in  $\mathbb{N}$  be such that  $f^N(v) < w$  and  $g^M(u) > w$ ; see Figure 1.1. We let  $\preceq_u$  be any induced ordering on  $G$  with  $u$  as first reference point. In particular,  $g^M \succ_u id$  and  $f^N g^M \succ_u id$ . We claim that  $\preceq_u$  is not Conradian. Indeed, we have that  $(f^N g^M)g^{2M} \prec_u g^M$ , since  $w < g^M(u) < v$  and  $f^M g^{3N}(u) < f^M(v) < w$ .  $\square$

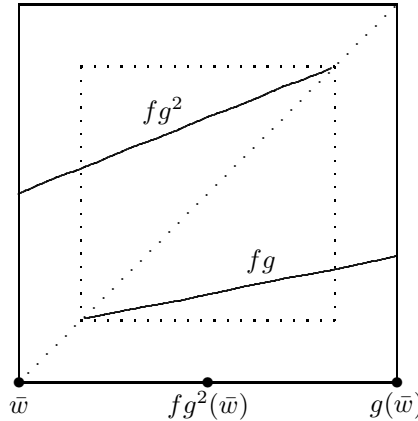


Figure 2.1: The crossing

As an application of Theorem A and/or Lemma 2.1.1, we give a new proof of a theorem proved in [35].

**Corollary 2.1.2.** *Let  $\preceq$  be an ordering on a group  $G$ . Let  $H$  be a subgroup of finite index such that  $\preceq$  restricted to  $H$  is Conradian. Then  $\preceq$  is a  $\mathcal{C}$ -ordering.*

*Proof:* Suppose, by way of a contradiction, that  $\preceq$  is a non Conradian ordering of  $G$ . Then, there are  $f, g$  in  $G$ , both  $\preceq$ -positive, such that  $fg^n \prec g$  for all  $n \in \mathbb{N}$ . By (the proof of) Theorem A,

$(f, g, id, f^{-1}g, g^2)$  is a crossing for the left translation action of  $G$  on itself. But then, for any  $n \in \mathbb{N}$ , we have that  $(f^n, g^n, id, f^{-n}g^n, g^{2n})$  is still a crossing for the action (see for instance Figure 2.2 below). But this is a contradiction, since, for certain  $n$  big enough,  $f^n$  and  $g^n$  belongs to  $H$ , thus  $(f^n, g^n, id, f^{-n}g^n, g^{2n})$  is a crossing for the left translation action of  $H$  on itself, which, by Theorem A, implies that the restriction of  $\preceq$  to  $H$  is non Conradian, contrary to our assumption.  $\square$

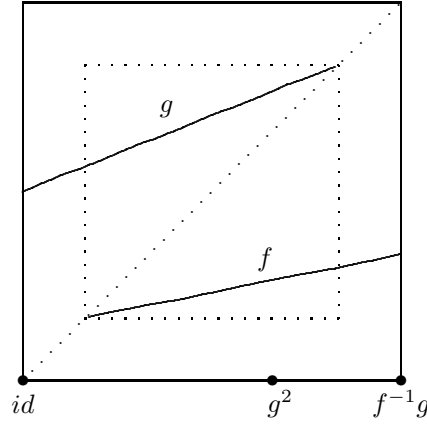


Figure 2.2

Note that if in (the proof of) Lemma 2.1.1 we let  $w_0$  be the smallest element (with respect to  $\leq^*$ ) of  $\Omega$ , then the stabilizer of  $w_0$  is  $\preceq$ -convex. Indeed, if  $id \prec g \prec f$ , with  $f(w_0) = w_0$ , then  $w_0 <^* w_f \leq^* w_g$ , thus  $g(w_0) = w_0$ . Actually, it is not hard to see that the same argument shows the following.

**Proposition 2.1.3.** *Let  $\Omega$  be a set endowed with a well-order  $\leq^*$ . If a group  $G$  acts faithfully on  $\Omega$  preserving a total order on it, then there exists a left-ordering on  $G$  for which the stabilizer  $G_{\Omega_0}$  of any initial segment  $\Omega_0$  of  $\Omega$  (w.r.t.  $\leq^*$ ) is convex. Moreover, if the action has no crossing, then this ordering is Conradian.*

**Example 2.1.4.** A very useful example of an action without crossings is the action by left translations on the set of left-cosets of any subgroup  $H$  which is convex with respect to a  $\mathcal{C}$ -ordering  $\preceq$  on  $G$ . Indeed, it is not hard to see that, due to the convexity of  $H$ , the order  $\preceq$  induces a total order  $\preceq_H$  on the set of left-cosets  $G/H$ . Moreover,  $\preceq_H$  is  $G$ -invariant. Now suppose that  $(f, g, uH, vH, wH)$  is a crossing for the action. Since  $w_1H \prec_H w_2H$  implies  $w_1 \prec w_2$ , for all  $w_1, w_2$  in  $G$ , we have that  $(f, g, u, v, w)$  is actually a crossing for the action by left translations of  $G$  on itself. Nevertheless, this contradicts Theorem A.

The following is an application of the preceding example. For the statement, we will say that a subgroup  $H$  of a group  $G$  is  $\mathcal{C}$ -relatively convex if there exists a  $\mathcal{C}$ -ordering on  $G$  for which  $H$  is convex.

**Lemma 2.1.5.** *For every  $\mathcal{C}$ -orderable group, the intersection of any family of  $\mathcal{C}$ -relatively convex subgroups is  $\mathcal{C}$ -relatively convex.*

*Proof:* We consider the action of  $G$  by left multiplications on each coordinate of the set  $\Omega = \prod_{\alpha} G/H_{\alpha}$ , where  $(G/H_{\alpha}, \preceq_{H_{\alpha}})$  is the  $(G$ -invariant ordered) set of left-cosets of the  $\mathcal{C}$ -relatively convex subgroup  $H_{\alpha}$ . Putting the (left) lexicographic order on  $\Omega$  and using Example 2.1.4, it is not hard to see that this action has no crossing. Moreover, since  $\{id\}$  is  $\mathcal{C}$ -convex, the action is faithful.

Now consider an arbitrary family  $\Omega_0 \subset \{H_\alpha\}_\alpha$  of  $\mathcal{C}$ -relatively convex subgroups of  $G$ , and let  $\leq^*$  be a well-order on  $\Omega$  for which  $\Omega_0$  is an initial segment. For the induced ordering  $\preceq$  on  $G$ , it follows from Proposition 2.1.3 that the stabilizer  $G_{\Omega_0} = \bigcap_{H \in \Omega_0} H$  is  $\preceq$ -convex. Moreover, Lemma 2.1.1 implies that  $\preceq$  is a  $\mathcal{C}$ -ordering, thus concluding the proof.  $\square$

We close this section with a simple lemma that we will need later and which may be left as an exercise to the reader (see also [20, Lemma 5.2.3]).

**Lemma 2.1.6.** *Let  $G$  be a torsion-free Abelian group. Then  $G$  admits only finitely many  $\mathcal{C}$ -orderings if and only if  $G$  has rank-one.*

### 2.1.2 Proof of Theorem B

Let  $G$  be a  $\mathcal{C}$ -orderable group admitting only finitely many  $\mathcal{C}$ -orderings. Obviously, each of these orderings must be isolated in  $\mathcal{CO}(G)$ . We claim that, in general, if  $\preceq$  is an isolated  $\mathcal{C}$ -ordering, then the series of  $\preceq$ -convex subgroups

$$\{id\} = G^{id} \subset \dots \subset G_g \triangleleft G^g \subset \dots \subset G$$

is finite. Indeed, let  $\{f_1, \dots, f_n\} \subset G$  be a set of  $\preceq$ -positive elements such that  $V_{f_1, \dots, f_n}$  consists only of  $\preceq$ . If the series above is infinite, then there exists  $g \in G$  so that no  $f_i$  belongs to  $G^g \setminus G_g$ . This implies that the flipped ordering  $\preceq^g$ , defined in Example 1.2.3, is Conradian and different from  $\preceq$ . However, every  $f_i$  is still  $\preceq^g$ -positive, which is impossible because  $V_{f_1, \dots, f_n} = \{\preceq\}$ .

Next let

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

be the series of  $\preceq$ -convex subgroups of  $G$ . According to Theorem 1.0.1, every quotient  $G_i/G_{i-1}$  embeds into  $\mathbb{R}$ , and thus it is Abelian. Since every ordering on such a quotient can be extended to an ordering on  $G$  (see for instance Corollary 1.2.5), the Abelian quotient  $G_i/G_{i-1}$  has only a finite number of orderings. It now follows from Lemma 2.1.6 that it must have rank one. Therefore, the series above is rational.

We now show that this series is unique. Suppose

$$\{id\} = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_k = G$$

is another rational series. Since  $H_{k-1}$  is  $\mathcal{C}$ -relatively convex, we conclude that  $N = G_{n-1} \cap H_{k-1}$  is  $\mathcal{C}$ -relatively convex by Lemma 2.1.5. Now  $G/N$  is torsion-free Abelian and has only a finite number of orderings, thus it has rank one. But  $H_{k-1}$  and  $G_{n-1}$  have the property that  $w^n \in G_{n-1}$  (resp.  $w^n \in H_{k-1}$ ) implies  $w \in G_{n-1}$  (resp.  $w \in H_{k-1}$ ). This clearly yields  $H_{k-1} = G_{n-1}$ . Repeating this argument several times, we conclude that rational series is unique, hence normal.

Now we claim that no quotient  $G_{i+2}/G_i$  is Abelian. If not,  $G_{i+2}/G_i$  would be a rank-two Abelian group, and so an infinite number of orderings could be defined on it. But since every ordering on this quotient can be extended to a  $\mathcal{C}$ -ordering on  $G$ , this would lead to a contradiction.

We now prove the converse of Theorem B.

Suppose that  $G$  has a normal rational series

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

such that no quotient  $G_{i+2}/G_i$  is Abelian. Clearly, flipping the orderings on the quotients  $G_{i+1}/G_i$  we obtain at least  $2^n$  many  $\mathcal{C}$ -orderings on  $G$ . We claim that these are the only possible  $\mathcal{C}$ -orderings



on  $G$ . Indeed, let  $\preceq$  be a  $\mathcal{C}$ -ordering on  $G$ , and let  $a \in G_1$  and  $b \in G_2 \setminus G_1$  be two non-commuting elements. Denoting the Conrad homomorphism of the group  $\langle a, b \rangle$  endowed with the restriction of  $\preceq$  by  $\tau$ , we have  $\tau(a) = \tau(bab^{-1})$ . Since  $G_1$  is rank-one Abelian, we have  $bab^{-1} = a^r$  for some rational number  $r \neq 1$ . Thus  $\tau(a) = r\tau(a)$ , which implies that  $\tau(a) = 0$ . Since  $\tau(|b|) > 0$ , where  $|b| = \max\{b^{-1}, b\}$ , we have that  $a^n \prec |b|$  for every  $n \in \mathbb{Z}$ . Since  $G_2/G_1$  is rank-one, this actually holds for every  $b \neq id$  in  $G_2 \setminus G_1$ . Thus  $G_1$  is convex in  $G_2$ .

Repeating the argument above, though now with  $G_{i+1}/G_i$  and  $G_{i+2}/G_i$  instead of  $G_1$  and  $G_2$ , respectively, we see that the rational series we began with is none other than the series given by the convex subgroups of  $\preceq$ . Since each  $G_{i+1}/G_i$  is rank-one Abelian, if we choose  $b_i \in G_{i+1} \setminus G_i$  for each  $i = 0, \dots, n-1$ , then any  $\mathcal{C}$ -ordering on  $G$  is completely determined by the signs of these elements. This shows that  $G$  admits precisely  $2^n$  different  $\mathcal{C}$ -orderings.

### 2.1.3 An example of a group with $2^n$ Conradian orderings but infinitely many left-orderings

The classification of groups having finitely many left-orderings, here stated as Theorem 1.1.1, was obtained by Tararin and appears in [20, §5.2]. In any of those groups, the number of left-orderings is  $2^n$  for some  $n \in \mathbb{N}$ . An example of a group having precisely  $2^n$  orders is  $T_n = \mathbb{Z}^n$  endowed with the product rule

$$(\alpha_n, \dots, \alpha_1) \cdot (\alpha'_n, \dots, \alpha'_1) = (\alpha_n + \alpha'_n, (-1)^{\alpha'_n} \alpha_{n-1} + \alpha'_{n-1}, \dots, (-1)^{\alpha'_2} \alpha_1 + \alpha'_1).$$

A presentation for  $T_n$  is  $\langle a_n, \dots, a_1 \mid R_n \rangle$ , where the set of relations  $R_n$  is

$$a_{i+1}a_i a_{i+1}^{-1} = a_i^{-1} \quad \text{if } i < n, \quad \text{and} \quad a_i a_j = a_j a_i \quad \text{if } |i - j| \geq 2.$$

Since the action of a group with only finitely many left-orderings on its corresponding space of left-orderings has only periodic orbits, Proposition 1.3.2 implies

**Corollary 2.1.7.** *Any left-ordering on a group with only finitely many left-orderings is Conradian.*

Therefore, it is natural to ask whether for each  $n \geq 2$  there are groups having precisely  $2^n$  Conradian orderings but infinitely many left-orderings. As it was shown in [36], for  $n = 2$  this is the case of the Baumslag-Solitar group  $B(1, \ell)$ ,  $\ell \geq 2$ ; see also §3.1. It turns out that, in order to construct examples for larger  $n$  and having  $B(1, \ell)$  as a quotient by a normal convex subgroup, we need to choose an odd integer  $\ell$ . As a concrete example, for  $n \geq 1$ , we endow  $C_n = \mathbb{Z} \times \mathbb{Z}[\frac{1}{3}] \times \mathbb{Z}^n$  (where  $\mathbb{Z}[\frac{1}{3}]$  denotes the group of triadic rational numbers) with the multiplication

$$\begin{aligned} \left( \gamma, \frac{\eta}{3^\kappa}, \alpha_n, \dots, \alpha_1 \right) \cdot \left( \gamma', \frac{\eta'}{3^{\kappa'}}, \alpha'_n, \dots, \alpha'_1 \right) = \\ = \left( \gamma + \gamma', 3^{-\gamma'} \frac{\eta}{3^\kappa} + \frac{\eta'}{3^{\kappa'}}, (-1)^{\eta'} \alpha_n + \alpha'_n, (-1)^{\alpha'_n} \alpha_{n-1} + \alpha'_{n-1}, \dots, (-1)^{\alpha'_2} \alpha_1 + \alpha'_1 \right), \end{aligned}$$

Note that the product rule is well defined because if  $\eta/3^\kappa = \bar{\eta}/3^{\bar{\kappa}}$ , then  $(-1)^\eta = (-1)^{\bar{\eta}}$  (it is here where we use the fact that  $\ell = 3$  is odd).

**Lemma 2.1.8.** *The group  $C_n$  admits the presentation*

$$C_n \cong \langle c, b, a_n, \dots, a_1 \mid cbc^{-1} = b^3, ca_i = a_i c, ba_n b^{-1} = a_n^{-1}, ba_i = a_i b \text{ if } i \neq n, R_n \rangle,$$

where  $R_n$  is the set of relations of  $T_n$  above.

*Proof:* Let  $\tilde{C}_n$  be the group with presentation  $\langle c, b, a_n, \dots, a_1 \mid cbc^{-1} = b^3, ca_i = a_ic, ba_nb^{-1} = a_n^{-1}, ba_i = a_ib \text{ if } i \neq n, R_n \rangle$ . Let  $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in C_n$ , where 1 is in the  $i$ -th vector entrance from the right, that is,  $e_1 = (0, \dots, 0, 1)$ . Clearly,  $\{e_i\}_{i=1}^{n+2}$  is a generating set for  $C_n$ . A direct calculation shows that

$$e_{i+1}e_ie_{i+1}^{-1} = e_i^{-1} \quad \text{if } i \leq n, \quad \text{and} \quad e_ie_j = e_je_i \quad \text{if } |i-j| \geq 2, \quad 1 \leq i \leq j \leq n+2.$$

Moreover,

$$e_{n+2}e_{n+1}e_{n+2}^{-1} = e_{n+1}^3, \quad e_{n+2}e_i = e_ie_{n+2} \quad \text{if } i \leq n \quad \text{and} \quad e_{n+1}e_i = e_ie_{n+1} \quad \text{if } i < n.$$

This means that  $\psi : \tilde{C}_n \rightarrow C_n$  defined by  $\psi(c) = e_{n+2}$ ,  $\psi(b) = e_{n+1}$ , and  $\psi(a_i) = e_i$  for  $1 \leq i \leq n$ , is a surjective homomorphism.

We now let  $\varphi : C_n \rightarrow \tilde{C}_n$  defined by  $\varphi((\gamma, \frac{\eta}{3^\kappa}, \alpha_n, \dots, \alpha_1)) = c^\gamma(c^{-\kappa}b^\eta c^\kappa)a_n^{\alpha_n} \dots a_1^{\alpha_1}$ , where we are assuming that  $\eta \in \mathbb{Z}$ . Note that, if  $\eta/3^\kappa = \bar{\eta}/3^{\bar{\kappa}}$ , where both  $\eta$  and  $\bar{\eta}$  are integers, then  $c^{-\kappa}b^\eta c^\kappa = c^{-\bar{\kappa}}b^{\bar{\eta}} c^{\bar{\kappa}}$ . For instance, if  $\kappa \leq \bar{\kappa}$ , then  $c^{\bar{\kappa}-\kappa}b^\eta c^{-\bar{\kappa}+\kappa} = b^{3^{\bar{\kappa}-\kappa}\eta} = b^{\bar{\eta}}$ . Therefore,  $\varphi$  is a well-defined function. To check that  $\varphi$  is a homomorphism, let  $\omega = (\gamma, \frac{\eta}{3^\kappa}, \alpha_n, \dots, \alpha_1)$  and  $\omega' = (\gamma', \frac{\eta'}{3^{\kappa'}}, \alpha'_n, \dots, \alpha'_1)$ . If  $\kappa + \gamma'$  and  $\kappa'$  are positive, then we have

$$\begin{aligned} \varphi(\omega)\varphi(\omega') &= c^\gamma(c^{-\kappa}b^\eta c^\kappa)a_n^{\alpha_n} \dots a_1^{\alpha_1} c^{\gamma'}(c^{-\kappa'}b^{\eta'} c^{\kappa'})a_n^{\alpha'_n} \dots a_1^{\alpha'_1} \\ &= c^{\gamma+\gamma'}(c^{-\kappa-\gamma'}b^\eta c^{\kappa+\gamma'})(c^{-\kappa'}b^{\eta'} c^{\kappa'})a_n^{(-1)^{\eta'}\alpha_n} a_{n-1}^{\alpha_{n-1}} \dots a_1^{\alpha_1} a_n^{\alpha'_n} \dots a_1^{\alpha'_1} \\ &= c^{\gamma+\gamma'}(c^{-\kappa-\gamma'-\kappa'}b^{3^{\kappa'}\eta+3^{\kappa+\gamma'}\eta'} c^{\kappa+\gamma'+\kappa'})a_n^{(-1)^{\eta'}\alpha_n+\alpha'_n} a_{n-1}^{(-1)^{\alpha'_n}\alpha_{n-1}+\alpha'_{n-1}} \dots a_1^{(-1)^{\alpha'_1}\alpha_1+\alpha'_1} \\ &= \varphi(\gamma+\gamma', \frac{3^{\kappa'}\eta+3^{\kappa+\gamma'}\eta'}{3^{\kappa+\gamma'+\kappa'}}, (-1)^{\eta'}\alpha_n+\alpha'_n, (-1)^{\alpha'_n}\alpha_{n-1}+\alpha'_{n-1}, \dots, (-1)^{\alpha'_1}\alpha_1+\alpha'_1) \\ &= \varphi(\omega\omega'). \end{aligned}$$

The equality  $\varphi(\omega)\varphi(\omega') = \varphi(\omega\omega')$  in the other cases can be checked similarly. Therefore,  $\varphi$  is a surjective homomorphism. Moreover,  $\varphi \circ \psi$  is the identity of  $\tilde{C}_n$  and  $\psi \circ \varphi$  is the identity of  $C_n$ , thus  $\varphi$  and  $\psi$  are isomorphisms. This finishes the proof of the lemma.  $\square$

The group  $C_n$  satisfies the hypotheses of Theorem B and has exactly  $2^{n+2}$  Conradian orderings. Indeed, for  $1 \leq i \leq n$  we let  $G^{(i)} = \langle e_1, \dots, e_i \rangle \triangleleft C_n$ . Note that  $C_n/G^{(n)} \simeq B(1, 3) = \langle f, g \mid fgf^{-1} = g^3 \rangle$ . We let  $G^{(n+1)}$  be the inverse image of the derived subgroup  $(C_n/G^{(n)})'$  under the projection  $C_n \rightarrow C_n/G^{(n)}$ . Clearly,  $C_n/G^{(n+1)} \simeq \mathbb{Z}$ , and  $G^{(n+1)}/G^{(n)} \simeq \mathbb{Z}[\frac{1}{3}]$ . Moreover, if we let  $G^{(n+2)} = C_n$  then, for  $1 \leq i \leq n+1$ , each quotient  $G^{(i+1)}/G^{(i)}$  is rank-one Abelian. Therefore, the series

$$\{id\} \triangleleft G^{(1)} \triangleleft \dots \triangleleft G^{(n+1)} \triangleleft G^{(n+2)} = C_n,$$

is rational. Finally, we have that, for  $1 \leq i \leq n$ ,  $G^{(i+2)}/G^{(i)}$  is non Abelian. Thus, the group  $C_n$  fits in the classification of groups with only finitely many  $\mathcal{C}$ -orderings. Nevertheless,  $C_n$  has  $B(1, 3)$  as a quotient by a normal convex subgroup. Since  $B(1, 3)$  admits uncountably many left-orderings, the same is true for  $C_n$ . In fact, it will follow from Theorem D that no left-ordering on  $C_n$  is isolated.

## 2.2 A structure theorem for the space of Conradian orderings

As shown by Linnell in [23], the space of left-orderings of a group is either finite or uncountable; see also §2.3 or [7, 33]. Although this is no longer true for bi-orderings, as there are examples of

groups having (only) infinitely countably many bi-orderings; see [2, §6.2], [5]. As announced in the Introduction, in this section we show

**Theorem C.** *Let  $G$  be a  $\mathcal{C}$ -orderable group. If  $G$  admits infinitely many  $\mathcal{C}$ -orderings, then it has uncountably many  $\mathcal{C}$ -orderings. Moreover, none of these is isolated in the space of  $\mathcal{C}$ -orderings.*

Note that the second statement implies that, if  $G$  is countable and admits infinitely many  $\mathcal{C}$ -orderings, then its space of Conradian orderings is a Cantor set. Note also that for the case of left-orderings there are group admitting infinitely many left-orderings together with isolated left-orderings, as it is the case of braid groups [11, 13] and the central extensions of Hecke groups appearing in [28].

### 2.2.1 Finitely many or a Cantor set of Conradian orders

Let  $G$  be a group admitting a  $\mathcal{C}$ -ordering  $\preceq$  that is isolated in the space of  $\mathcal{C}$ -orderings. As we have seen at the beginning of §2.1.2, the series of  $\preceq$ -convex subgroups must be finite, say

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G.$$

Proceeding as in Example 1.2.3, any ordering on  $G_{i+1}/G_i$  may be extended (preserving the set of positive elements outside of it) to a  $\mathcal{C}$ -ordering on  $G$ . Hence, each quotient must be rank-one Abelian, so the series above is rational. We claim that this series of  $\preceq$ -convex subgroups is unique (hence normal) and that no quotient  $G_{i+2}/G_i$  is Abelian. In fact, if the series has length 2, then it is normal. Moreover, since no  $\mathcal{C}$ -ordering on a rank-two Abelian group is isolated, we have that  $G_2$  is non Abelian. Then, by Theorem B, the series is unique. In the general case, we will use induction on the length of the series. Suppose that every group having an isolated  $\mathcal{C}$ -ordering whose rational series of convex subgroups

$$\{id\} = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_k$$

has length  $k < n$  admits a unique (hence normal) rational series and that no quotient  $H_{i+2}/H_i$  is Abelian. Let

$$\{id\} = G_0 \triangleleft \dots \triangleleft G_{n-2} \triangleleft G_{n-1} \triangleleft G_n = G$$

be a rational series of length  $n$  associated to some isolated  $\mathcal{C}$ -ordering  $\preceq$  on  $G$ . Since  $G_{n-1}$  is normal in  $G$ , for every  $g \in G$ , the conjugate series

$$\{id\} = G_0^g \triangleleft \dots \triangleleft G_{n-2}^g \triangleleft G_{n-1}^g = G_{n-1}$$

is also a rational series for  $G_{n-1}$ . Since this series is associated to a certain isolated  $\mathcal{C}$ -ordering, namely the restriction of  $\preceq$  to  $G_{n-1}$ , we conclude that it is unique by the induction hypothesis. Hence the series must coincide with the original one, or in other words  $G_i^g = G_i$ . Therefore, the series for  $G$  is normal. Moreover, every quotient  $G_{i+2}/G_i$  is non Abelian, because if not then  $\preceq$  could be approximated by other  $\mathcal{C}$ -orderings on  $G$ . Thus, by Theorem B, the rational series for  $G$  is unique, and  $G$  admits only finitely many  $\mathcal{C}$ -orderings. This completes the proof of Theorem C.

### 2.2.2 An application to orderable nilpotent groups

In this section we use Theorem C to give a new proof of the following result which was first proved in [31].

**Theorem 2.2.1 (Navas).** *Let  $G$  be a torsion-free nilpotent group which is not rank-one Abelian. Then the space of left-orderings of  $G$  is homeomorphic to a Cantor set.*

The proof of Theorem 2.2.1 is a consequence of three facts.

The first one is that torsion-free nilpotent groups are left-orderable. Indeed, as shown in [22, §2.6], they admit a filtration

$$\{id\} \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G,$$

such that each quotient  $G_i/G_{i-1}$  is torsion-free Abelian (actually they are bi-orderable). In particular they are  $\mathcal{C}$ -orderable. Furthermore, torsion-free nilpotent groups have the (much!) stronger property that every partial left-ordering can be extended to a (total) left-ordering; see [2, Theorem 7.6.4].

The second fact is a result shown independently by Ault and Rhemtulla, and appears for instance in [2, §7.5]. For the convenience of the reader we give a short proof of this fact:

**Proposition 2.2.2.** *Every left-ordering on a nilpotent group is Conradian.*

*Proof:* Let  $(G, \preceq)$  be a left-ordered nilpotent group. We claim that the action of  $G$  on itself has no crossings.

Suppose, by way of a contradiction, that  $(f, g, u, v, w)$  is a crossing. Then, by definition, we have that  $f^N(v) \prec w \prec g^M(u)$ . Then, a classical ping-pong argument shows that  $\{f^N, g^M\}$  generates a free semigroup. But this is impossible since, as it is well known, nilpotent groups can not have free semigroups (one may think, for instance, in the growth rate of the subgroup  $\langle f^N, g^M \rangle$ , see also [29, exercise 4.47]).  $\square$

The third fact is that the only nilpotent,  $\mathcal{C}$ -orderable group with finitely many (Conradian) orderings are the rank-one Abelian groups. To see this is enough to note that the groups described in Theorem B, whose rational series has length 2 or more, have trivial center.

This finishes the proof of Theorem 2.2.1.

**Remark 2.2.3.** A direct consequence of the proof of Proposition 2.2.2, is that any left-ordering on a group without free semigroups on two generators is Conradian. In particular,  $H$ , the Grigorchuk-Machi group of intermediate growth from [16] (see also [30]), is a group admitting only Conradian orderings. Moreover, from the natural action of  $H$  on  $\mathbb{Z}^{\mathbb{N}}$ , one can be induced infinitely many left-orderings on it. Therefore, the space of left-orderings of  $H$  is a Cantor set.

## 2.3 A structure theorem for left-orderings

As announced in the Introduction, in this section we explore the possibility of approximating a given left-ordering by its conjugates (in the sense of §1.3.1). We will show that in most cases this can be done. This will give a new proof of Linnell's result from [23], here stated as Theorem E.

### 2.3.1 Describing the Conradian soul via crossings

The *Conradian soul*  $C_{\preceq}(G)$  of an ordered group  $(G, \preceq)$  corresponds to the maximal (with respect to the inclusion) subgroup which is convex in  $\preceq$ , and such that  $\preceq$  restricted to the subgroup is Conradian. This notion was introduced in [31], where a dynamical counterpart in the case of countable groups was given. To give an analogous characterization in the general case, we consider the set  $S^+$  formed by the elements  $h \succ id$  such that  $h \preceq w$  for every crossing  $(f, g, u, v, w)$  satisfying  $id \preceq u$ . Analogously, we let  $S^-$  be the set formed by the elements  $h \prec id$  such that  $w \preceq h$  for every crossing  $(f, g, u, v, w)$  satisfying  $v \preceq id$ . Finally, we let

$$S = \{id\} \cup S^+ \cup S^-.$$

*A priori*, it is not clear that the set  $S$  has a nice structure (for instance, it is not at all evident that it is actually a subgroup). However, this is largely shown by the next theorem.

**Theorem 2.3.1.** *The Conradian soul of  $(G, \preceq)$  coincides with the set  $S$  above.*

Before passing to the proof, we give four general lemmata describing the flexibility of the concept of crossings for group orderings (note that the first three lemmata still apply to crossings for actions on totally ordered spaces). The first one allows us replacing the “comparison” element  $w$  by its “images” under positive iterates of either  $f$  or  $g$ .

**Lemma 2.3.2.** *If  $(f, g, u, v, w)$  is a crossing, then  $(f, g, u, v, g^n w)$  and  $(f, g, u, v, f^n w)$  are also crossings for every  $n \in \mathbb{N}$ .*

*Proof:* We will only consider the first 5-uple (the case of the second one is analogous). Recalling that  $gw \succ w$ , for every  $n \in \mathbb{N}$  we have  $u \prec w \prec g^n w$ ; moreover,  $v \succ g^{M+n}u = g^n g^M u \succ g^n w$ . Hence,  $u \prec g^n w \prec v$ . On the other hand,  $f^N v \prec w \prec g^n w$ , while from  $g^M u \succ w$  we get  $g^{M+n}u \succ g^n w$ .  $\square$

Our second lemma allows replacing the “limiting” elements  $u$  and  $v$  by more appropriate ones.

**Lemma 2.3.3.** *Let  $(f, g, u, v, w)$  be a crossing. If  $fu \succ u$  (resp.  $fu \prec u$ ) then  $(f, g, f^n u, v, w)$  (resp.  $(f, g, f^{-n}u, v, w)$ ) is also a crossing for every  $n > 0$ . Analogously, if  $gv \prec v$  (resp.  $gv \succ v$ ), then  $(f, g, u, g^n v, w)$  (resp.  $(f, g, u, g^{-n}v, w)$ ) is also crossing for every  $n > 0$ .*

*Proof:* Let us only consider the first 5-uple (the case of the second one is analogous). Suppose that  $fu \succ u$  (the case  $fu \prec u$  may be treated similarly). Then  $f^n u \succ u$ , which gives  $g^M f^n u \succ g^M u \succ w$ . To show that  $f^n u \prec w$ , assume by contradiction that  $f^n u \succeq w$ . Then  $f^n u \succ f^N v$ , which yields  $u \succ f^{N-n}v$ , which is absurd.  $\square$

The third lemma relies on the dynamical insight of the crossing condition.

**Lemma 2.3.4.** *If  $(f, g, u, v, w)$  is a crossing, then  $(hfh^{-1}, hgh^{-1}, hu, hv, hw)$  is also a crossing for every  $h \in G$ .*

*Proof:* The three conditions to be checked are nothing but the three conditions in the definition of crossing multiplied by  $h$  by the left.  $\square$

A direct application of the lemma above shows that, if  $(f, g, u, v, w)$  is a crossing, then the 5-uples  $(f, f^n g f^{-n}, f^n u, f^n v, f^n w)$  and  $(g^n f g^{-n}, g, g^n u, g^n v, g^n w)$  are also crossings for every  $n \in \mathbb{N}$ . This combined with Lemma 2.3.3 may be used to show the following.

**Lemma 2.3.5.** *If  $(f, g, u, v, w)$  is a crossing and  $id \preceq h_1 \prec h_2$  are elements in  $G$  such that  $h_1 \in S$  and  $h_2 \notin S$ , then there exists a crossing  $(f, \tilde{g}, \tilde{u}, \tilde{v}, \tilde{w})$  such that  $h_1 \prec \tilde{u} \prec \tilde{v} \prec h_2$ .*

*Proof:* Since  $id \prec h_2 \notin S$ , there must be a crossing  $(f, g, u, v, w)$  such that  $id \preceq u \prec w \prec h_2$ . Let  $N \in \mathbb{N}$  be such that  $f^N v \prec w$ . Denote by  $(f, \bar{g}, \bar{u}, \bar{v}, \bar{w})$  the crossing  $(f, f^N g f^{-N}, f^N u, f^N v, f^N w)$ . Note that  $\bar{v} = f^N v \prec w \prec h_2$ . We claim that  $h_1 \preceq \bar{w} = f^N w$ . Indeed, if  $f^N u \succ u$ , then  $f^N u \succ id$ , and by the definition of  $S$  we must have  $h_1 \preceq \bar{w}$ . If  $f^N u \prec u$ , then we must have  $fu \prec u$ , so by Lemma 2.3.3 we know that  $(f, \bar{g}, u, \bar{v}, \bar{w})$  is also a crossing, which allows still concluding that  $h_1 \preceq \bar{w}$ .

Now for the crossing  $(f, \bar{g}, \bar{u}, \bar{v}, \bar{w})$  there exists  $M \in \mathbb{N}$  such that  $\bar{w} \prec \bar{g}^M \bar{u}$ . Let us consider the crossing  $(\bar{g}^M f \bar{g}^{-M}, \bar{g}, \bar{g}^M \bar{u}, \bar{g}^M \bar{v}, \bar{g}^M \bar{w})$ . If  $\bar{g}^M \bar{v} \prec \bar{v}$ , then  $\bar{g}^M \bar{v} \prec h_2$ , and we are done. If not, then

we must have  $\bar{g}\bar{v} \succ \bar{v}$ . By Lemma 2.3.3,  $(\bar{g}^M f \bar{g}^{-M}, \bar{g}, \bar{g}^M \bar{u}, \bar{g}^M \bar{v}, \bar{w})$  is still a crossing, and since  $\bar{v} \prec h_2$ , this concludes the proof.  $\square$

*Proof of Theorem 2.3.1.* The proof is divided into several steps.

Claim 0. The set  $S$  is convex.

This follows directly from the definition of  $S$ .

Claim 1. If  $h$  belongs to  $S$ , then  $h^{-1}$  also belongs to  $S$ .

Assume that  $h \in S$  is positive and  $h^{-1}$  does not belong to  $S$ . Then there exists a crossing  $(f, g, u, v, w)$  so that  $h^{-1} \prec w \prec v \preceq id$ .

We first note that, if  $h^{-1} \preceq u$ , then after conjugating by  $h$  as in Lemma 2.3.4, we get a contradiction because  $(hgh^{-1}, hfh^{-1}, hu, hv, hw)$  is a crossing with  $id \preceq hu$  and  $hw \prec hv \preceq h$ . To reduce the case  $h^{-1} \succ u$  to this one, we first use Lemma 2.3.4 and we consider the crossing  $(g^M f g^{-M}, g, g^M u, g^M v, g^M w)$ . Since  $h^{-1} \prec w \prec g^M u \prec g^M w \prec g^M v$ , if  $g^M v \prec v$  then we are done. If not, Lemma 2.3.3 shows that  $(g^M f g^{-M}, g, g^M u, g^M v, w)$  is also a crossing, which still allows concluding.

In the case where  $h \in S$  is negative we proceed similarly but we conjugate by  $f^N$  instead of  $g^M$ . Alternatively, since  $id \in S$  and  $id \prec h^{-1}$ , if we suppose that  $h^{-1} \notin S$  then Lemma 2.3.5 provides us with a crossing  $(f, g, u, v, w)$  such that  $id \prec u \prec w \prec v \prec h^{-1}$ , which gives a contradiction after conjugating by  $h$ .

Claim 2. If  $h$  and  $\bar{h}$  belong to  $S$ , then  $h\bar{h}$  also belongs to  $S$ .

First we show that for every positive elements in  $S$ , their product still belongs to  $S$ . (Note that, by Claim 1, the same will be true for products of negative elements in  $S$ .) Indeed, suppose that  $h, \bar{h}$  are positive elements, with  $h \in S$  but  $h\bar{h} \notin S$ . Then, by Lemma 2.3.5 we may produce a crossing  $(f, g, u, v, w)$  such that  $h \prec u \prec v \prec h\bar{h}$ . After conjugating by  $h^{-1}$  we obtain the crossing  $(h^{-1}fh, h^{-1}gh, h^{-1}u, h^{-1}v, h^{-1}w)$  satisfying  $id \prec h^{-1}u \prec h^{-1}v \prec \bar{h}$ , which shows that  $\bar{h} \notin S$ .

Now, if  $h \prec id \prec \bar{h}$ , then  $h \prec h\bar{h}$ . Hence, if  $h\bar{h}$  is negative, then the convexity of  $S$  gives  $h\bar{h} \in S$ . If  $h\bar{h}$  is positive, then  $\bar{h}^{-1}h^{-1}$  is negative, and since  $\bar{h}^{-1} \prec \bar{h}^{-1}h^{-1}$ , the convexity gives again that  $\bar{h}^{-1}h^{-1}$ , and hence  $h\bar{h}$ , belongs to  $S$ . The remaining case  $\bar{h} \prec id \prec h$  may be treated similarly.

Claim 3. The subgroup  $S$  is Conradian.

In order to apply Theorem A, we need to show that there are no crossings in  $S$ . Suppose by contradiction that  $(f, g, u, v, w)$  is a crossing such that  $f, g, u, v, w$  all belong to  $S$ . If  $id \preceq w$ , then, by Lemma 2.3.4, we have that  $(g^n f g^{-n}, g, g^n u, g^n v, g^n w)$  is a crossing. Taking  $n = M$  so that  $g^M u \succ w$ , this gives a contradiction to the definition of  $S$  because  $id \preceq w \prec g^M u \prec g^M w \prec g^M v \in S$ . The case  $w \preceq id$  may be treated in an analogous way by conjugating by powers of  $f$  instead of  $g$ .

Claim 4. The subgroup  $S$  is maximal among  $\preceq$ -convex,  $\preceq$ -Conradian subgroups.

Indeed, if  $C$  is a subgroup strictly containing  $S$ , then there is a positive  $h$  in  $C \setminus S$ . By Lemma 2.3.5, there exists a crossing  $(f, g, u, v, w)$  such that  $id \prec u \prec w \prec v \prec h$ . If  $C$  is convex, then  $u, v, w$  belong to  $C$ . To conclude that  $C$  is not Conradian, it suffices to show that  $f$  and  $g$  belong to  $C$ .

Since  $id \prec u$ , we have either  $id \prec g \prec gu \prec v$  or  $id \prec g^{-1} \prec g^{-1}u \prec v$ . In both cases, the convexity of  $C$  implies that  $g$  belongs to  $C$ . On the other hand, if  $f$  is positive, then from  $f^N \prec f^N v \prec w$  we get  $f \in C$ , whereas in the case of a negative  $f$  the inequality  $id \prec u$  gives  $id \prec f^{-1} \prec f^{-1}u \prec v$ , which still shows that  $f \in C$ .  $\square$

### 2.3.2 Approximating a left-orderings by its conjugates

Recall from §1.3 that the *positive cone* of a left-ordering  $\preceq$  in  $\mathcal{LO}(G)$  is the set  $P$  of its positive elements. Because of the left invariance,  $P$  completely determines  $\preceq$ . The *conjugate* of  $\preceq$  by  $h \in G$  is the left-ordering  $\preceq_h$  having positive cone  $hPh^{-1}$ . In other words,  $g \succ_h id$  holds if and only if  $hgh^{-1} \succ id$ . We will say that  $\preceq$  may be approximated by its conjugates if it is an accumulation point of its set of conjugates.

**Theorem 2.3.6.** *Suppose  $(G, \preceq)$  is a non trivial left-ordered group such that it has trivial Conradian soul. Then  $\preceq$  may be approximated by its conjugates.*

*Proof:* Let  $f_1 \prec f_2 \prec \dots \prec f_k$  be finitely many positive elements in  $G$ . We need to show that there exists a conjugate of  $\preceq$  that is different from  $\preceq$  but for which all the  $f_i$ 's are still positive. Since  $id \in C_{\preceq}(G)$  and  $f_1 \notin C_{\preceq}(G)$ , Theorem 2.3.1 and Lemma 2.3.5 imply that there is a crossing  $(f, g, u, v, w)$  such that  $id \prec u \prec v \prec f_1$ . Let  $M, N$  in  $\mathbb{N}$  be such that  $f^N v \prec w \prec g^M u$ . We claim that  $id \prec_{v^{-1}} f_i$  and  $id \prec_{w^{-1}} f_i$  for  $1 \leq i \leq k$ , but  $g^M f^N \prec_{v^{-1}} id$  and  $g^M f^N \succ_{w^{-1}} id$ . Indeed, since  $id \prec v \prec f_i$ , we have  $v \prec f_i \prec f_i v$ , thus  $id \prec v^{-1} f_i v$ . By definition, this means that  $f_i \succ_{v^{-1}} id$ . The inequality  $f_i \succ_{w^{-1}} id$  is proved similarly. Now note that  $g^M f^N v \prec g^M w \prec v$ , and so  $g^M f^N \prec_{v^{-1}} id$ . Finally, from  $g^M f^N w \succ g^M u \succ w$  we get  $g^M f^N \succ_{w^{-1}} id$ .

Now the preceding relations imply that the  $f_i$ 's are still positive for both  $\preceq_{v^{-1}}$  and  $\preceq_{w^{-1}}$ , but at least one of these left-orderings is different from  $\preceq$ . This concludes the proof.  $\square$

Based on the work of Linnell [23], it is shown in [31, Proposition 4.1] that no Conradian ordering is an isolated point of the space of left-orderings of a group having infinitely many left-orderings. Note that this result also follows as a combination of Theorem C and Theorem D. Together with Theorem 2.3.6, this shows the next proposition by means of the convex extension procedure (c.f., Corollary 1.2.5).

**Proposition 2.3.7.** *Let  $G$  be a left-orderable group. If  $\preceq$  is an isolated point of  $\mathcal{LO}(G)$ , then its Conradian soul is nontrivial and has only finitely many left-orderings.*

As a consequence of Tararin's theorem, here Theorem 1.1.1, the number of left-orderings on a left-orderable group having only finitely many left-orderings is a power of 2. Moreover, all of these left-orderings are necessarily Conradian; see Corollary 2.1.7. By the preceding theorem, if  $\preceq$  is an isolated point of the space of left-orderings  $\mathcal{LO}(G)$ , then its Conradian soul admits  $2^n$  different left-orderings for some  $n \geq 1$ , all of them Conradian. Let  $\{\preceq_1, \preceq_2, \dots, \preceq_{2^n}\}$  be these left-orderings, where  $\preceq_1$  is the restriction of  $\preceq$  to its Conradian soul. Since  $C_{\preceq}(G)$  is  $\preceq$ -convex, each  $\preceq_j$  induces a left-ordering  $\preceq^j$  on  $G$ , namely the convex extension of  $\preceq_j$  by  $\preceq$ . (Note that  $\preceq^1$  coincides with  $\preceq$ .) The Lemma below appears in [31]. For the reader's convenience we provide a proof of this fact.

**Lemma 2.3.8.** *With the notations above, all the left-orderings  $\preceq^j$  share the same Conradian soul.*

*Proof:* Consider the left-ordering  $\preceq^j$ . Since  $\preceq^j$  restricted to  $C_{\preceq}(G)$  is Conradian, and  $C_{\preceq}(G)$  is convex in  $\preceq^j$ , we only need to check that  $C_{\preceq^j}(G) \subseteq C_{\preceq}(G)$ . Let  $G^*$  be any  $\preceq^j$ -convex subgroup strictly containing  $C_{\preceq}(G)$ . We claim that  $G^*$  is also  $\preceq$ -convex. Indeed, since  $\preceq^j$  coincides with  $\preceq$  outside  $C_{\preceq}(G)$ , we have that for any  $f \notin C_{\preceq}(G)$ ,  $id \prec f$  if and only if  $id \prec^j f$ ; see for instance §1.2. In particular,  $id \prec h \prec g$  for  $g \in G^*$  and  $g \notin C_{\preceq}(G)$ , implies  $h \prec^j g$ , hence  $h \in G^*$ , and the claim follows.

Since  $G^*$  is  $\preceq$ -convex and strictly contains  $C_{\preceq}(G)$ , we have that there are  $f, g$  in  $G^*$  such that  $id \prec f \prec g$  and  $fg^n \prec g$  for all  $n \in \mathbb{N}$ . Clearly  $g \notin C_{\preceq}(G)$ . We claim that for all  $n \in \mathbb{N}$ , the

element  $g^{-1}fg^n$  does not belong to  $C_{\preceq}(G)$ . Indeed, if  $g^{-1}fg^n \in C_{\preceq}(G)$ , then  $(g^{-1}fg^n)^{-1} \prec g$ , which implies that  $id \prec gfg^{n+1}$  contrary to our choice of  $f$  and  $g$ .

If it is the case that  $f \notin C_{\preceq}(G)$ , then we are done. Indeed, since  $id \prec^j f$ ,  $id \prec^j g$  and  $g^{-1}fg^n \prec^j id$  for all  $n \in \mathbb{N}$ , we have that  $\preceq^j$  restricted to  $G^*$  is not Conradian. In the case that  $f \in C_{\preceq}(G)$ , we let  $h = fg$ . Note that  $h \notin C_{\preceq}(G)$  and that  $h \succ^j id$ . Moreover, as before, we have that  $g^{-1}hg^n \prec id$  for all  $n \in \mathbb{N}$  and  $g^{-1}hg^n \notin C_{\prec}(G)$ . This shows that  $\preceq^j$  restricted to  $G^*$  is not Conradian.  $\square$

Below, assume that  $\preceq$  is not Conradian.

**Theorem 2.3.9.** *With the notation above, at least one of the left-orderings  $\preceq^j$  is an accumulation point of the set of conjugates of  $\preceq$ .*

Before proving this theorem, we immediately state

**Corollary 2.3.10.** *At least one of the left-orderings  $\preceq^j$  is approximated by its conjugates.*

*Proof:* Assuming Theorem 2.3.9, we have  $\preceq^k \in \text{acc}(\text{orb}(\preceq^1))$  for some  $k \in \{1, \dots, 2^n\}$ . Theorem 2.3.9 applied to this  $\preceq^k$  instead of  $\preceq$  shows the existence of  $k' \in \{1, \dots, 2^n\}$  so that  $\preceq^{k'} \in \text{acc}(\text{orb}(\preceq^k))$ , and hence  $\preceq^{k'} \in \text{acc}(\text{orb}(\preceq^1))$ . If  $k'$  equals either 1 or  $k$  then we are done; if not, we continue arguing in this way... In at most  $2^n$  steps we will find an index  $j$  such that  $\preceq^j \in \text{acc}(\text{orb}(\preceq^1))$ .  $\square$

Theorem 2.3.9 will follow from the next

**Proposition 2.3.11.** *Given an arbitrary finite family  $\mathcal{G}$  of  $\preceq$ -positive elements in  $G$ , there exists  $h \in G$  and  $id \prec \bar{h} \notin C_{\preceq}(G)$  such that  $id \prec h^{-1}fh \notin C_{\preceq}(G)$  for all  $f \in \mathcal{G} \setminus C_{\preceq}(G)$ , but  $id \succ h^{-1}\bar{h}h \notin C_{\preceq}(G)$ .*

*Proof of Theorem 2.3.9 from Proposition 2.3.11:* Let us consider the directed set formed by the finite sets  $\mathcal{G}$  of  $\preceq$ -positive elements. For each such a  $\mathcal{G}$ , let  $h_{\mathcal{G}}$  and  $\bar{h}_{\mathcal{G}}$  be the elements in  $G$  provided by Proposition 2.3.11. After passing to subnets of  $(h_{\mathcal{G}})$  and  $(\bar{h}_{\mathcal{G}})$  if necessary, we may assume that the restrictions of  $\preceq_{h_{\mathcal{G}}^{-1}}$  to  $C_{\preceq}(G)$  all coincide with a single  $\preceq_j$ . Now the properties of  $h_{\mathcal{G}}$  and  $\bar{h}_{\mathcal{G}}$  imply:

- $f \succ^j id$  and  $f(\succ^j)_{h_{\mathcal{G}}^{-1}} id$  for all  $f \in \mathcal{G} \setminus C_{\preceq}(G)$ ,
- $\bar{h}_{\mathcal{G}} \succ^j id$ , but  $\bar{h}_{\mathcal{G}}(\prec^j)_{h_{\mathcal{G}}^{-1}} id$ .

This shows Theorem 2.3.9.  $\square$

For the proof of Proposition 2.3.11 we will use three general lemmata.

**Lemma 2.3.12.** *For every  $id \prec c \notin C_{\preceq}(G)$  there is a crossing  $(f, g, u, v, w)$  such that  $u, v, w$  do not belong to  $C_{\preceq}(G)$  and  $id \prec u \prec w \prec v \prec c$ .*

*Proof:* By Theorem 2.3.1 and Lemma 2.3.5, for every  $id \preceq s \in C_{\preceq}(G)$  there exists a crossing  $(f, g, u, v, w)$  such that  $s \prec u \prec w \prec v \prec c$ . Clearly,  $v$  does not belong to  $C_{\preceq}(G)$ . The element  $w$  is also outside  $C_{\preceq}(G)$ , since in the other case the element  $a = w^2$  would satisfy  $w \prec a \in C_{\preceq}(G)$ , which is absurd. Taking  $M > 0$  so that  $g^M u \succ w$ , this gives  $g^M u \notin C_{\preceq}(G)$ ,  $g^M w \notin C_{\preceq}(G)$ , and  $g^M v \notin C_{\preceq}(G)$ . Consider the crossing  $(g^M f g^{-M}, g, g^M u, g^M v, g^M w)$ . If  $g^M v \prec v$ , then we are done. If not, then  $gv \succ v$ , and Lemma 2.3.3 ensures that  $(g^M f g^{-M}, g, g^M u, v, g^M w)$  is also a crossing, which still allows concluding.  $\square$

**Lemma 2.3.13.** *Given  $id \prec c \notin C_{\preceq}(G)$  there exists  $id \prec a \notin C_{\preceq}(G)$  (with  $a \prec c$ ) such that, for all  $id \preceq b \preceq a$  and all  $\bar{c} \succeq c$ , one has  $id \prec b^{-1}\bar{c}b \notin C_{\preceq}(G)$ .*



*Proof:* Let us consider the crossing  $(f, g, u, v, w)$  such that  $id \prec u \prec w \prec v \prec c$  and such that  $u, v, w$  do not belong to  $C_{\preceq}(G)$ . We affirm that the lemma holds for  $a = u$  (actually, it holds for  $a = w$ , but the proof is slightly more complicated). Indeed, if  $id \preceq b \preceq u$ , then from  $b \preceq u \prec v \prec \bar{c}$  we get  $id \preceq b^{-1}u \prec b^{-1}v \prec b^{-1}\bar{c}$ , and thus the crossing  $(b^{-1}fb, b^{-1}gb, b^{-1}u, b^{-1}v, b^{-1}w)$  shows that  $b^{-1}\bar{c} \notin C_{\preceq}(G)$ . Since  $id \preceq b$ , we conclude that  $id \prec b^{-1}\bar{c} \preceq b^{-1}\bar{c}b$ , and the convexity of  $S$  implies that  $b^{-1}\bar{c}b \notin C_{\preceq}(G)$ .  $\square$

**Lemma 2.3.14.** *For every  $g \in G$  the set  $gC_{\preceq}(G)$  is convex. Moreover, for every crossing  $(f, g, u, v, w)$  one has  $uC_{\preceq}(G) \prec wC_{\preceq}(G) \prec vC_{\preceq}(G)$ , in the sense that  $uh_1 \prec wh_2 \prec vh_3$  for all  $h_1, h_2, h_3$  in  $C_{\preceq}(G)$ .*

*Proof:* The verification of the convexity of  $gC_{\preceq}(G)$  is straightforward. Now suppose that  $uh_1 \succ wh_2$  for some  $h_1, h_2$  in  $C_{\preceq}(G)$ . Then since  $u \prec w$ , the convexity of both left classes  $uC_{\preceq}(G)$  and  $wC_{\preceq}(G)$  gives the equality between them. In particular, there exists  $h \in C_{\preceq}(G)$  such that  $uh = w$ . Note that such an  $h$  must be positive, so that  $id \prec h = u^{-1}w$ . But since  $(u^{-1}fu, u^{-1}gu, id, u^{-1}v, u^{-1}w)$  is a crossing, this contradicts the definition of  $C_{\preceq}(G)$ . Showing that  $wC_{\preceq}(G) \prec vC_{\preceq}(G)$  is similar.  $\square$

*Proof of Proposition 2.3.11:* Let us label the elements of  $\mathcal{G} = \{f_1, \dots, f_r\}$  so that  $f_1 \prec \dots \prec f_r$ , and let  $k$  be such that  $f_{k-1} \in C_{\preceq}(G)$  but  $f_k \notin C_{\preceq}(G)$ . Recall that, by Lemma 2.3.13, there exists  $id \prec a \notin C_{\preceq}(G)$  such that, for every  $id \preceq b \preceq a$ , one has  $id \prec b^{-1}f_{k+j}b \notin C_{\preceq}(G)$  for all  $j \geq 0$ . We fix a crossing  $(f, g, u, v, w)$  such that  $id \prec u \prec v \prec a$  and  $u \notin C_{\preceq}(G)$ . Note that the conjugacy by  $w^{-1}$  gives the crossing  $(w^{-1}fw, w^{-1}gw, w^{-1}u, w^{-1}v, id)$ .

Case 1. One has  $w^{-1}v \preceq a$ .

In this case, the proposition holds for  $h = w^{-1}v$  and  $\bar{h} = w^{-1}g^{M+1}f^Nw$ . To show this, first note that neither  $w^{-1}gw$  nor  $w^{-1}fw$  belong to  $C_{\preceq}(G)$ . Indeed, this follows from the convexity of  $C_{\preceq}(G)$  and the inequalities  $w^{-1}g^{-M}w \prec w^{-1}u \notin C_{\preceq}(G)$  and  $w^{-1}f^{-N}w \succ w^{-1}v \notin C_{\preceq}(G)$ . We also have  $id \prec w^{-1}g^Mf^Nw$ , and hence  $id \prec w^{-1}gw \prec w^{-1}g^{M+1}f^Nw$ , which shows that  $\bar{h} \notin C_{\preceq}(G)$ . On the other hand, the inequality  $w^{-1}g^{M+1}f^Nw(w^{-1}v) \prec w^{-1}v$  reads as  $h^{-1}\bar{h}h \prec id$ . Finally, Lemma 2.3.2 applied to the crossing  $(w^{-1}fw, w^{-1}gw, w^{-1}u, w^{-1}v, id)$  shows that  $(w^{-1}fw, w^{-1}gw, w^{-1}u, w^{-1}v, w^{-1}g^{M+n}f^Nw)$  is a crossing for every  $n > 0$ . For  $n \geq M$  we have  $w^{-1}g^{M+1}f^Nw(w^{-1}v) \prec w^{-1}g^{M+n}f^Nw$ . Since  $w^{-1}g^{M+n}f^Nw \prec w^{-1}v$ , Lemma 2.3.14 easily implies that  $w^{-1}g^{M+1}f^Nw(w^{-1}v)C_{\preceq}(G) \prec w^{-1}vC_{\preceq}(G)$ , that is,  $h^{-1}\bar{h}h \notin C_{\preceq}(G)$ .

Case 2. One has  $a \prec w^{-1}v$ , but  $w^{-1}g^mw \preceq a$  for all  $m > 0$ .

We claim that, in this case, the proposition holds for  $h = a$  and  $\bar{h} = w^{-1}g^{M+1}f^Nw$ . This may be checked in the very same way as in Case 1 by noticing that, if  $a \prec w^{-1}v$  but  $w^{-1}g^mw \preceq a$  for all  $m > 0$ , then  $(w^{-1}fw, w^{-1}gw, w^{-1}u, a, id)$  is a crossing.

Case 3. One has  $a \prec w^{-1}v$  and  $w^{-1}g^mw \succ a$  for some  $m > 0$ . (Note that the first condition follows from the second one.)

We claim that, in this case, the proposition holds for  $h = a$  and  $\bar{h} = w \notin C_{\preceq}(G)$ . Indeed, we have  $g^mw \succ ha$  (and  $w \prec ha$ ), and since  $g^mw \prec v \prec a$ , we have  $wa \prec a$ , which means that  $h^{-1}\bar{h}h \prec id$ . Finally, from Lemmas 2.3.2 and 2.3.14 we get  $waC_{\preceq}(G) \preceq g^mwC_{\preceq}(G) \prec vC_{\preceq}(G) \preceq aC_{\preceq}(G)$ . This implies that  $a^{-1}waC_{\preceq}(G) \prec C_{\preceq}(G)$ , which means that  $h^{-1}\bar{h}h \notin C_{\preceq}(G)$ .  $\square$

### 2.3.3 Finitely many or uncountably many left-orderings

The goal of this final short section is to use the previously developed ideas to give an alternative proof of the following result due to Linnell; see [23].

**Theorem E (Linnell).** *If the space of orderings of an orderable group is infinite, then it is uncountable.*

*Proof:* Let us fix an ordering  $\preceq$  on an orderable group  $G$ . We need to analyze two different cases.

Case 1. The Conradian soul of  $C_{\preceq}(G)$  is nontrivial and has infinitely many left-orderings.

This case was settled in [31] (see Proposition 4.1 therein) using ideas going back to Zenkov [41]. Alternatively we can use Theorem C and D to conclude that  $C_{\preceq}(G)$  has no isolated left-orderings, so it is uncountable. By proposition 1.2.1, the same is true for the space of left-orderings of  $G$ .

Case 2. The Conradian soul of  $C_{\preceq}(G)$  has only finitely many orderings.

If  $\preceq$  is Conradian, then  $G = C_{\preceq}(G)$  has finitely many orderings. If not, then Theorems 2.3.6 and 2.3.9 imply that there exists an ordering  $\preceq^*$  on  $G$  which is an accumulation point of its conjugates. The closure in  $\mathcal{LO}(G)$  of the set of conjugates of  $\preceq^*$  is then a compact set without isolated points. By a well-known fact in General Topology, such a set must be uncountable. Therefore,  $G$  admits uncountably many orderings.  $\square$

## Chapter 3

# Left-orders on groups with finitely many $\mathcal{C}$ -orders

The main result of this chapter is motivated by the following

**Question 3.0.15.** Is it true that for left-orderable, solvable groups, having an isolated left-ordering is equivalent to having only finitely many left-orderings?

Indeed, as far as the author knows, the only examples of groups having infinitely many left-orderings together with isolated left-orderings are braid groups, [11, 13], and the groups introduced in [28]. Both families of groups are not solvable (actually they contain free subgroups). On the other hand, the dichotomy holds for nilpotent groups; see Theorem 2.2.1.

Here we focus on a (very) restricted subfamily of solvable groups, namely, groups having only finitely many Conradian orderings. These groups are described in Theorem B. We show

**Theorem D.** *If a  $\mathcal{C}$ -orderable group has only finitely many  $\mathcal{C}$ -orderings, then its space of left-orderings is either finite or homeomorphic to the Cantor set.*

As shown in Theorem B, a group with finitely many Conradian orderings admits a unique rational series. Therefore, it is countable, so  $\mathcal{LO}(G)$  is metrizable. Thus, in order to prove Theorem D, we need to show that no left-ordering of  $G$  is isolated, unless there are only finitely many of them.

We proceed by induction on the length of the rational series. In §3.1 we explore the case  $n = 2$ . In this case we will give an explicit description of  $\mathcal{LO}(G)$ . This extends [36], where the space of orderings of the Baumslag-Solitar  $B(1, \ell)$ ,  $\ell \geq 2$ , is described. In §3.2.1, we obtain some technical lemmata partially describing the inner automorphisms of a group with a finite number of Conradian orderings. As a result we show that the maximal convex subgroup of  $G$  (with respect to a  $\mathcal{C}$ -ordering) is a group that fits into the classification made by Tararin, *i.e.* a group with only finitely many left-orderings (a *Tararin group*, for short); see Theorem 1.1.1. Finally, in §3.2.2, we prove the inductive step. Section 3.2.3 is devoted to the description of an illustrative example.

### 3.1 The metabelian case

Throughout this section,  $G$  will denote a left-orderable, non Abelian group with rational series of length 2:

$$\{id\} = G_0 \triangleleft G_1 \triangleleft G_2 = G.$$

If  $G$  is not bi-orderable, then for the rational series above the quotient  $G_2/G_0 = G$  is non bi-orderable. Therefore,  $G$  fits into the classification made by Tararin, here Theorem 1.1.1, so it has only finitely many left-orderings.

For the rest of this section, we will assume that  $G$  is not a Tararin group, hence  $G$  is bi-orderable. We have

**Lemma 3.1.1.** *The group  $G$  satisfies  $G/G_1 \simeq \mathbb{Z}$ .*

*Proof:* Indeed, consider the action by conjugation  $\alpha : G/G_1 \rightarrow \text{Aut}(G_1)$  given by  $\alpha(gG_1)(h) = ghg^{-1}$ . Since  $G$  is non Abelian, we have that this action is nontrivial, i.e.  $\text{Ker}(\alpha) \neq G/G_1$ . Moreover,  $\text{Ker}(\alpha) = \{id\}$ , since in the other case, as  $G/G_1$  is rank-one Abelian, we would have that  $(G/G_1)/\text{Ker}(\alpha)$  is a torsion group. But the only nontrivial, finite order automorphism of  $G_1$  is the inversion, which implies that  $G$  is non bi-orderable, thus a Tararin group.

The following claim is elementary and it we leave its proof to the reader.

Claim. If  $G$  is a torsion-free, rank-one Abelian group such that  $G \not\simeq \mathbb{Z}$ , then for any  $g \in G$ , there is an integer  $n > 1$  and  $g_n \in G$  such that  $g_n^n = g$ .

Now take any  $b \in G \setminus G_1$  so that  $\alpha(bG_1)$  is a nontrivial automorphism of  $G_1$ . Since  $G_1$  is rank-one Abelian, for some positive  $r = p/q \in \mathbb{Q}$ ,  $r \neq 1$ , we must have that  $bab^{-1} = a^r$  for all  $a \in G_1$ . Suppose that  $G/G_1 \not\simeq \mathbb{Z}$ . By the previous claim, we have a sequence of increasing integers  $(n_1, n_2, \dots)$  and a sequence  $(g_1, g_2, \dots)$  of elements in  $G/G_1$  such that  $g_i^{n_i} = bG_1$ . But clearly this can not happen since for  $g_i$  we have that  $g_i a g_i^{-1} = a^{r_i}$ , where  $r_i$  is a rational such that  $r_i^{n_i} = r$ , which is impossible. (In other words, given  $r$  we have found, among the rational numbers, an infinite collection of  $r_i$  solving the equation  $x^{n_i} - r = 0$ , but, by the Rational Roots Theorem or Rational Roots Test this can not happen; see for instance [26, Proposition 5.1].) This finishes the proof of Lemma 1.1.  $\square$

**Lemma 3.1.2.** *The group  $G$  embeds in  $Af_+(\mathbb{R})$ , the group of (orientation-preserving) affine homeomorphism of the real line.*

*Proof:* We first embed  $G_1$ . Fix  $a \in G_1$ ,  $a \neq id$ . Define  $\varphi_a : G_1 \rightarrow Af_+(\mathbb{R})$  by declaring  $\varphi_a(a)(x) = x + 1$ , and if  $a' \in G_1$  is such that  $(a')^q = a^p$ , we declare  $\varphi_a(a')(x) = x + p/q$ . Showing that  $\varphi_a$  is an injective homomorphism is routine.

Now let  $b \in G$  be such that  $\langle bG_1 \rangle = G/G_1$ . Let  $1 \neq r \in \mathbb{Q}$  such that  $ba'b^{-1} = (a')^r$  for every  $a' \in G_1$ . Since  $G$  is bi-orderable we have that  $r > 0$ , and changing  $b$  by  $b^{-1}$  if necessary, we may assume that  $r > 1$ . Then, given  $w \in G$ , there is a unique  $n \in \mathbb{Z}$  and a unique  $\bar{w} \in G_1$  such that  $w = b^n \bar{w}$ .

Define  $\varphi_{b,a} : G \rightarrow Af_+(\mathbb{R})$  by letting  $\varphi_{b,a}(b^n \bar{w}) = H_r^{(n)} \circ \varphi_a(\bar{w})$ , where  $H_r(x) = rx$ , and  $H_r^{(n)}$  is the  $n$ -th iterate of  $H_r$  (by convention  $H_r^{(0)}(x) = x$ ). We claim that  $\varphi_{b,a}$  is an injective homomorphism.

Indeed, let  $w_1, w_2$  in  $G$ ,  $w_1 = b^{n_1} \bar{w}_1$ ,  $w_2 = b^{n_2} \bar{w}_2$ . Let  $r_1 \in \mathbb{Q}$  be such that  $\varphi_a(\bar{w}_1)(x) = x + r_1$ . Then  $H_r^{(n)} \circ \varphi_a(b^{-n} \bar{w}_1 b^n)(x) = H_r^{(n)} \circ \varphi_a(\bar{w}_1^{(1/r)^n})(x) = r^n(x + (1/r)^n r_1) = \varphi_a(\bar{w}_1) \circ H_r^{(n)}(x)$ , for all  $n \in \mathbb{Z}$ . Thus

$$\begin{aligned} \varphi_{b,a}(w_1 w_2) &= \varphi_{b,a}(b^{n_1} b^{n_2} b^{-n_2} \bar{w}_1 b^{n_2} \bar{w}_2) = H_r^{(n_1)} \circ H_r^{(n_2)} \circ \varphi_a(b^{-n_2} \bar{w}_1 b^{n_2}) \circ \varphi_a(\bar{w}_2) \\ &= H_r^{(n_1)} \circ \varphi_a(\bar{w}_1) \circ H_r^{(n_2)} \circ \varphi_a(\bar{w}_2) = \varphi_{b,a}(w_1) \circ \varphi_{b,a}(w_2), \end{aligned}$$

which shows that  $\varphi$  is a homomorphism. To see that it is injective, suppose that  $\varphi(w_1)(x) = \varphi(b^{n_1} \bar{w}_1)(x) = r^{n_1} x + r^{n_1} r_1 = x$  for all  $x \in \mathbb{R}$ . Then  $n = 0$  and  $r_1 = 0$ , showing that  $w_1 = id$ . This finishes the proof of Lemma 1.2  $\square$

Once the embedding  $\varphi = \varphi_{b,a} : G \rightarrow Af_+(\mathbb{R})$  is fixed, we can associate to each irrational number  $\varepsilon$  an *induced left-ordering*  $\preceq_\varepsilon$  on  $G$  whose set of positive elements is  $\{g \in G \mid \varphi(g)(\varepsilon) > \varepsilon\}$ . When  $\varepsilon$  is rational, the preceding set defines only a partial ordering. However, in this case the stabilizer of the point  $\varepsilon$  is isomorphic to  $\mathbb{Z}$ , hence this partial ordering may be completed to two total left-orderings  $\preceq_\varepsilon^+$  and  $\preceq_\varepsilon^-$ . These orderings were introduced by Smirnov in [38]. Once the representation  $\varphi$  is fixed, we call these orderings, together with its corresponding reverse orderings, Smirnov-type orderings.

Besides the Smirnov-type orderings on  $G$ , there are four Conradian (actually bi-invariant) orderings. Since  $G_1$  is always convex in a Conradian ordering, the sign of  $b^n a^s \in G$ ,  $n \neq 0$ , depends only on the sign of  $b$  and  $n$ . Then it is not hard to check that the four Conradian orderings are the following:

- 1)  $\preceq_{C_1}$ , defined by  $id \prec_{C_1} b^n a^s$  ( $n \in \mathbb{Z}$ ,  $s \in \mathbb{Q}$ ) if and only if either  $n \geq 1$ , or  $n = 0$  and  $s > 0$ .
- 2)  $\preceq_{C_2}$ , defined by  $id \prec_{C_2} b^n a^s$  if and only if either  $n \leq -1$ , or  $n = 0$  and  $s > 0$ .
- 3)  $\preceq_{C_3} = \overline{\preceq}_{C_1}$  (the reverse ordering of  $\preceq_{C_1}$ ).
- 4)  $\preceq_{C_4} = \overline{\preceq}_{C_2}$ .

**Proposition 3.1.3.** *Let  $U \subseteq \mathcal{LO}(G)$  be the set consisting of the four Conradian orderings together with the Smirnov-type orderings. Then any ordering in  $U$  is non isolated in  $U$ .*

*Proof:* We first show that the Conradian orderings are non isolated.

Indeed, we claim that  $\preceq_\varepsilon \rightarrow \preceq_{C_1}$  as  $\varepsilon \rightarrow \infty$ . To show this, it suffices to show that any positive element in the  $\preceq_{C_1}$  ordering becomes  $\preceq_\varepsilon$ -positive for any  $\varepsilon$  large enough.

By definition of  $\preceq_\varepsilon$ , we have that  $id \prec_\varepsilon b^n a^s$  if and only if  $r^n(\varepsilon + s) = \varphi(b^n a^s)(\varepsilon) > \varepsilon$ , where  $r > 1$ . Now, assume that  $id \prec_{C_1} b^n a^s$ . If  $n = 0$ , then  $s > 0$  and  $\varepsilon + s > \varepsilon$ . If  $n \geq 1$ , then  $r^n(\varepsilon + s) > \varepsilon$  for  $\varepsilon > \frac{\varepsilon r^n s}{r^n - 1}$ . Thus the claim follows.

In order to approximate the other three Conradian orderings, we first note that, arguing just as before, we have  $\preceq_\varepsilon \rightarrow \preceq_{C_2}$  as  $\varepsilon \rightarrow -\infty$ . Finally, the other two Conradian orderings  $\overline{\preceq}_{C_1}$  and  $\overline{\preceq}_{C_2}$  are approximated by  $\overline{\preceq}_\varepsilon$  as  $\varepsilon \rightarrow \infty$  and  $\varepsilon \rightarrow -\infty$ , respectively.

Now let  $\preceq_S$  be an Smirnov-type ordering and let  $\{g_1, \dots, g_n\}$  be a set of  $\preceq_S$ -positive elements.

Suppose first that  $\preceq_S = \preceq_\varepsilon$ , where  $\varepsilon$  is irrational. Then we have that  $\varphi(g_i)(\varepsilon) > \varepsilon$  for all  $1 \leq i \leq n$ . Thus, if  $\varepsilon'$  is such that  $\varepsilon < \varepsilon' < \min\{\varphi(g_i)(\varepsilon)\}$ ,  $1 \leq i \leq n$ , then we still have that  $\varphi(g_i)(\varepsilon') > \varepsilon'$ , hence  $g_i \succ_{\varepsilon'} id$  for  $1 \leq i \leq n$ . To see that  $\succ_{\varepsilon'} \neq \succ_\varepsilon$ , first notice that  $\varphi(G_1)(x)$  is dense in  $\mathbb{R}$  for all  $x \in \mathbb{R}$ . In particular, taking  $g \in G_1$  such that  $\varepsilon < \varphi(g)(0) < \varepsilon'$ , we have that  $\varphi(g b^n g^{-1})(\varepsilon) = \varphi(g)(r^n \varphi(g)^{-1}(\varepsilon)) = r^n \varphi(g)^{-1}(\varepsilon) + \varphi(g)(0)$ . Since  $\varphi(g)^{-1}(\varepsilon) < 0$ , we have that for  $n$  large enough,  $g b^n g^{-1} \prec_\varepsilon id$ . The same argument shows that  $g b^n g^{-1} \succ_{\varepsilon'} id$ . Therefore,  $\preceq_{\varepsilon'}$  and  $\preceq_\varepsilon$  are distinct.

The remaining case is  $\preceq_S = \preceq_\varepsilon^\pm$ , where  $\varepsilon$  is rational. In this case we can order the set  $\{g_1, \dots, g_n\}$  in such a way that there is  $i_0$  with  $\varphi(g_i)(\varepsilon) > \varepsilon$  for  $1 \leq i \leq i_0$ , and  $\varphi(g_i)(\varepsilon) = \varepsilon$  for  $i_0 + 1 \leq i \leq n$ . That is,  $g_i \in \text{Stab}(\varepsilon) \simeq \mathbb{Z}$  for  $i_0 + 1 \leq i \leq n$ . Let  $\varepsilon' > \varepsilon$ .

We claim that either  $\varphi(g_i)(\varepsilon') > \varepsilon'$  for all  $i_0 + 1 \leq i \leq n$  or  $\varphi(g_i)(\varepsilon') < \varepsilon'$  for all  $i_0 + 1 \leq i \leq n$ . Indeed, since  $\varphi$  gives an affine action, it can not be the case that a nontrivial element of  $G$  fixes two points. Hence, we have that  $\varphi(g_i)(\varepsilon') \neq \varepsilon'$  for each  $i_0 + 1 \leq i \leq n$ . Now, suppose for a contradiction that there are  $g_{i_0}$ ,  $g_{i_1}$  in  $\text{Stab}(\varepsilon)$  with  $g_{i_0}(\varepsilon') < \varepsilon'$  and  $g_{i_1}(\varepsilon') > \varepsilon'$ . Let  $n, m$  in  $\mathbb{N}$  be such that  $g_{i_0}^n = g_{i_1}^m$ . Then  $\varepsilon' < \varphi(g_{i_1})^m(\varepsilon') = \varphi(g_{i_0})^n(\varepsilon') < \varepsilon'$ , which is a contradiction. Thus the claim follows.

Now assume that  $\varphi(g_i)(\varepsilon') > \varepsilon'$ , for all  $i_0 + 1 \leq i \leq n$ . If  $\varepsilon < \varepsilon' < \min\{\varphi(g_i)(\varepsilon)\}$ , with  $1 \leq i \leq i_0$ , then  $g_i \succ_{\varepsilon'} id$  for  $1 \leq i \leq n$ , showing that  $\preceq_S$  is non isolated. In the case where  $\varphi(g_i)(\varepsilon') < \varepsilon'$ , for all  $i_0 + 1 \leq i \leq n$ , we let  $\tilde{\varepsilon}$  be such that  $\max\{\varphi(g_i)^{-1}(\varepsilon)\} < \tilde{\varepsilon} < \varepsilon$  for  $1 \leq i \leq i_0$ . Then we have that  $g_i \succ_{\tilde{\varepsilon}} id$  for  $1 \leq i \leq n$ . This shows that, in any case,  $\preceq_S = \preceq_{\tilde{\varepsilon}}^{\pm}$  is non isolated in  $U$ .  $\square$

The following theorem shows that the space of left-orderings of  $G$  is made up by the Smirnov-type orderings together with the Conradian orderings. This generalizes [36, Theorem 1.2].

**Theorem 3.1.4.** *Suppose  $G$  is a non Abelian group with rational series of length 2. If  $G$  is bi-orderable, then its space of left-orderings has no isolated points. Moreover, every non-Conradian ordering is equal to an induced, Smirnov-type ordering, arising from the affine action of  $G$  on the real line given by  $\varphi$  above.*

To prove Theorem 3.1.4, we will use the ideas (and notation) involved in the *dynamical realization of an ordering*, here Proposition 1.4.2.

*Proof of Theorem 3.1.4:* First fix  $a \in G_1$  and  $b \in G$  exactly as above, that is, such that  $bab^{-1} = a^r$ , where  $r \in \mathbb{Q}$ ,  $r > 1$ , and  $\varphi(a)(x) = x + 1$ ,  $\varphi(b)(x) = rx$ . Now let  $\preceq$  be a left-ordering on  $G$ , and consider its dynamical realization. To prove Theorem 3.1.4, we will distinguish two cases:

**Case 1.** The element  $a \in G$  is cofinal (that is, for every  $g \in G$ , there are  $n_1, n_2$  in  $\mathbb{Z}$  such that  $a^{n_1} \prec g \prec a^{n_2}$ ).

Note that, in a Conradian ordering,  $G_1$  is convex, hence  $a$  cannot be cofinal. Thus, in this case we have to prove that  $\preceq$  is an Smirnov-type ordering.

For the next two claims, recall that for any measure  $\mu$  on a measurable space  $X$  and any measurable function  $f : X \rightarrow X$ , the *push-forward measure*  $f_*(\mu)$  is defined by  $f_*(\mu)(A) = \mu(f^{-1}(A))$ , where  $A \subseteq X$  is a measurable subset. Note that  $f_*(\mu)$  is trivial if and only if  $\mu$  is trivial. Moreover, one has  $(fg)_*(\mu) = f_*(g_*(\mu))$  for all measurable functions  $f, g$ .

Similarly, the *push-backward measure*  $f^*(\mu)$  is defined by  $f^*(\mu)(A) = \mu(f(A))$ .

Claim 1. The subgroup  $G_1$  preserves a Radon measure  $\nu$  (i.e., a measure that is finite on compact sets) on the real line which is unique up to a scalar multiplication and has no atoms.

Since  $a$  is cofinal and  $G_1$  is rank-one Abelian, its action on the real line is *free* (that is, no point is fixed by any nontrivial element of  $G_1$ ). By Hölder's theorem (see [14, Theorem 6.10] or [29, §2.2]), the action of  $G_1$  is semi-conjugated to a group of translations. More precisely, there exists a non-decreasing, continuous, surjective function  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  such that, to each  $g \in G_1$ , one may associate a translation parameter  $c_g$  so that, for all  $x \in \mathbb{R}$ ,

$$\rho(g(x)) = \rho(x) + c_g.$$

Now since the Lebesgue measure  $Leb$  on the real line is invariant under translations, the *push-backward measure*  $\nu = \rho^*(Leb)$  is invariant by  $G_1$ . Since  $Leb$  is a Radon measure without atoms, this is also the case of  $\nu$ .

To see the uniqueness of  $\nu$  up to scalar multiple, we follow [29, §2.2.5]. Given any  $G_1$ -invariant measure  $\mu$ , we consider the associated *translation number homomorphism*  $\tau_\mu : G_1 \rightarrow \mathbb{R}$  defined by

$$\tau_\mu(g) = \begin{cases} \mu([x, g(x)]) & \text{if } g(x) > x, \\ 0 & \text{if } g(x) = x, \\ -\mu([g(x), x]) & \text{if } g(x) < x. \end{cases}$$

One easily checks that this definition is independent of  $x \in \mathbb{R}$ , and that the kernel of  $\tau_\mu$  coincides with the set of elements having fixed points, which in this case reduces to the identity element of  $G_1$ . Now, from [29, Proposition 2.2.38], to prove the uniqueness of  $\nu$ , it is enough to show that, for any nontrivial Radon measure  $\mu$  invariant under the action of  $G_1$ ,  $\tau_\mu(G_1)$  is dense in  $\mathbb{R}$ . But since  $G_1$  is rank-one Abelian and  $G_1 \not\cong \mathbb{Z}$ , any nontrivial homomorphism from  $G_1$  to  $\mathbb{R}$  has a dense image. In particular  $\tau_\mu(G_1)$  is dense in  $\mathbb{R}$ . So Claim 1 follows.

**Claim 2.** For some  $\lambda \neq 1$ , we have  $b_*(\nu) = \lambda\nu$ .

Since  $G_1 \triangleleft G$ , for any  $a' \in G_1$  and all measurable  $A \subset \mathbb{R}$ , we must have

$$b_*(\nu)(a'(A)) = \nu(b^{-1}a'(A)) = \nu(\bar{a}(b^{-1}(A))) = \nu(b^{-1}(A)) = b_*(\nu)((A))$$

for some  $\bar{a} \in G_1$ . (Actually,  $a' = \bar{a}^r$ .) Thus  $b_*(\nu)$  is a measure that is invariant under  $G_1$ . The uniqueness of the  $G_1$ -invariant measure up to a scalar factor yields  $b_*(\nu) = \lambda\nu$  for some  $\lambda > 0$ . Assume for a contradiction that  $\lambda$  equals 1. Then the whole group  $G$  preserves  $\nu$ . In this case, the *translation number homomorphism* is defined on  $G$ . The kernel of  $\tau_\nu$  must contain the commutator subgroup of  $G$ , and, since  $a^{r-1} = [a, b] \in [G, G]$ , we have that  $\tau_\nu(a^{r-1}) = 0$ , hence  $\tau_\nu(a) = 0$ . Nevertheless, this is impossible, since the kernel of  $\tau_\nu$  coincides with the set of elements having fixed points on the real line (see [29, §2.2.5]). So Claim 2 is proved.

By Claims 1 and 2, for each  $g \in G$  we have  $g_*(\nu) = \lambda_g(\nu)$  for some  $\lambda_g > 0$ . Moreover,  $\lambda_a = 1$  and  $\lambda_b = \lambda \neq 1$ . Note that, since  $(fg)_*(\nu) = f_*(g_*(\nu))$ , the correspondence  $g \mapsto \lambda_g$  is a group homomorphism from  $G$  to  $\mathbb{R}_+$ , the group of positive real numbers under multiplication. Since  $G_1$  is in the kernel of this homomorphism and any  $g \in G$  is of the form  $b^n a^s$  for  $n \in \mathbb{Z}$ ,  $s \in \mathbb{Q}$ , we have that the kernel of this homomorphism is exactly  $G_1$ .

**Lemma 3.1.5.** Let  $A : G \rightarrow Af_+(\mathbb{R})$ ,  $g \mapsto A_g$ , be defined by

$$A_g(x) = \frac{1}{\lambda_g}x + \frac{\text{sgn}(g)}{\lambda_g}\nu([t(g^{-1}), t(id)]),$$

where  $\text{sgn}(g) = \pm 1$  is the sign of  $g$  in  $\preceq$ . Then  $A$  is an injective homomorphism.

*Proof:* For  $g, h$  in  $G$  both  $\preceq$ -positive, we compute

$$\begin{aligned} A_{gh}(x) &= \frac{1}{\lambda_{gh}}x + \frac{1}{\lambda_{gh}}\nu([t((gh)^{-1}), t(id)]) \\ &= \frac{1}{\lambda_g\lambda_h}x + \frac{1}{\lambda_g\lambda_h}[(h_*\nu)([t(g^{-1}), t(h)])] \\ &= \frac{1}{\lambda_g\lambda_h}x + \frac{1}{\lambda_g\lambda_h}[\lambda_h\nu([t(g^{-1}), t(id)]) + \nu([t(h^{-1}), t(id)])] \\ &= \frac{1}{\lambda_g\lambda_h}x + \frac{1}{\lambda_g}\nu([t(g^{-1}), t(id)]) + \frac{1}{\lambda_g\lambda_h}\nu([t(h^{-1}), t(id)]) \\ &= A_g(A_h(x)). \end{aligned}$$

The other cases can be treated analogously, thus showing that  $A$  is a group homomorphism.

Now, assume that  $A_g(x) = x$  for some nontrivial  $g \in G$ . Then  $\lambda_g = 1$ . In particular,  $g \in G_1$ , since the kernel of the application  $g \mapsto \lambda_g$  is  $G_1$ . But in this case we have that  $g$  has no fixed point, thus assuming that  $0 = \lambda_g^{-1}\nu([t(g^{-1}), t(id)]) = \nu([t(g^{-n}), t(id)])$  implies that  $\nu$  is the trivial (zero) measure. This contradiction settles Lemma 3.1.5.  $\square$

Now, for  $x \in \mathbb{R}$ , let  $F(x) = \text{sgn}(x - t(id)) \cdot \nu([t(id), x])$ . (Note that  $F(t(id)) = 0$ .) By semi-conjugating the dynamical realization by  $F$  we (re)obtain the faithful representation  $A : G \rightarrow Af_+(\mathbb{R})$ . More precisely, for all  $g \in G$  and all  $x \in \mathbb{R}$ , we have

$$F(g(x)) = A_g(F(x)). \quad (3.1)$$

For instance, if  $x > t(id)$  and  $g \succ id$ , then

$$\begin{aligned} F(g(x)) &= \nu([t(id), g(x)]) \\ &= \frac{1}{\lambda_g} \nu([t(g^{-1}), x]) \\ &= \frac{1}{\lambda_g} \nu([t(g^{-1}), t(id)]) + \frac{1}{\lambda_g} \nu([t(id), x]) \\ &= \frac{1}{\lambda_g} F(x) + \frac{1}{\lambda_g} \nu([t(g^{-1}), t(id)]). \end{aligned}$$

The action  $A$  induces a (perhaps partial) left-ordering  $\preceq_A$ , namely  $g \succ_A id$  if and only if  $A_g(0) > 0$ . Note that equation (3.1) implies that for every  $g \in G_1$  such that  $g \succ id$ , we have  $A_g(0) > 0$ , hence  $g \succ_A id$ . Similarly, for every  $f \in G$  such that  $A_f(0) > 0$ , we have  $f \succ id$ . In particular, if the orbit under  $A$  of 0 is free (that is, for every nontrivial element  $g \in G$ , we have  $A_g(0) \neq 0$ ), then (3.1) yields that  $\preceq_A$  is total and coincides with  $\preceq$  (our original ordering).

If the orbit of 0 is not free (this may arise for example when  $t(id)$  does not belong to the support of  $\nu$ ), then the stabilizer of 0 under the action of  $A$  is isomorphic to  $\mathbb{Z}$ . Therefore,  $\preceq$  coincides with either  $\preceq_A^+$  or  $\preceq_A^-$  (the definition of  $\preceq_A^\pm$  is similar to that of  $\preceq_\varepsilon^\pm$  above).

At this point we have that  $\preceq$  can be realized as an induced ordering from the action given by  $A$ . Therefore, arguing as in the proof of Proposition 3.1.3, we have that  $\preceq_A$ , hence  $\preceq$ , is non isolated.

To show that  $\preceq$  is an Smirnov-type ordering, we need to determine all possible embeddings of  $G$  into the affine group. Recall that  $bab^{-1} = a^r$ , where  $r = p/q > 1$ .

**Lemma 3.1.6.** *Every faithful representation of  $G$  in the affine group is given by*

$$a \sim \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} r & \beta \\ 0 & 1 \end{pmatrix}$$

for some  $\alpha \neq 0$  and  $\beta \in \mathbb{R}$ .

*Proof:* Arguing as in Lemma 3.1.2 one may check that  $\varphi'_{a,b} : \{a, b\} \rightarrow Af_+(\mathbb{R})$  defined by  $\varphi'_{a,b}(a)(x) = x + \alpha$  and  $\varphi'_{a,b}(b)(x) = rx + \beta$  may be (uniquely) extended to an homomorphic embedding  $\varphi'_{a,b} : G \rightarrow Af_+(\mathbb{R})$ . Conversely, let

$$a \sim \begin{pmatrix} s & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} t & \beta \\ 0 & 1 \end{pmatrix}$$

be a representation. Since we are dealing with orientation-preserving affine maps, both  $s$  and  $t$  are positive real numbers. Moreover, the following equality must hold:

$$a^p \sim \begin{pmatrix} s^p & s^{p-1}\alpha + \dots + s\alpha + \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s^q & s^{q-1}\alpha t + s^{q-2}\alpha t + \dots + \alpha t - s^q\beta + \beta \\ 0 & 1 \end{pmatrix} \sim ba^qb^{-1}.$$

Thus  $s = 1$  and  $t = p/q = r$ . Finally, since the representation is faithful,  $\alpha \neq 0$ .  $\square$



Let  $\alpha, \beta$  be such that  $A_a(x) = x + \alpha$  and  $A_b(x) = rx + \beta$ . We claim that if the stabilizer of 0 under  $A$  is trivial –which implies in particular that  $\beta \neq 0$ –, then  $\preceq_A$  (hence  $\preceq$ ) coincides with  $\preceq_\varepsilon$  if  $\alpha > 0$  (resp.  $\preceq_\varepsilon^-$  if  $\alpha < 0$ ), where  $\varepsilon = \frac{\beta}{(r-1)\alpha}$ . Indeed, if  $\alpha > 0$ , then for each  $g = b^n a^s \in G$ ,  $s \in \mathbb{Q}$ , we have  $A_g(0) = r^n s \alpha + \beta \frac{r^n - 1}{r - 1}$ . Hence  $A_g(0) > 0$  holds if and only if

$$r^n \frac{\beta}{(r-1)\alpha} + r^n s > \frac{\beta}{(r-1)\alpha}.$$

Letting  $\varepsilon = \frac{\beta}{(r-1)\alpha}$ , one easily checks that the preceding inequality is equivalent to  $g \succ_\varepsilon id$ . The claim now follows.

In the case where the stabilizer of 0 under  $A$  is isomorphic to  $\mathbb{Z}$ , similar arguments to those given above show that  $\preceq$  coincides with either  $\preceq_\varepsilon^+$ , or  $\preceq_\varepsilon^-$ , or  $\preceq_\varepsilon^+$ , or  $\preceq_\varepsilon^-$ , where  $\varepsilon$  again equals  $\frac{\beta}{(r-1)\alpha}$ .

**Case 2.** The element  $a \in G$  is not cofinal.

In this case, for the dynamical realization of  $\preceq$ , the set of fixed points of  $a$ , denoted  $Fix(a)$ , is non-empty. We claim that  $b(Fix(a)) = Fix(a)$ . Indeed, let  $r = p/q$ , and let  $x \in Fix(a)$ . We have

$$a^p(b(x)) = a^p b(x) = b a^q(x) = b(x).$$

Hence  $a^p(b(x)) = b(x)$ , which implies that  $a(b(x)) = b(x)$ , as asserted. Note that, since there is no global fixed point for the dynamical realization, we must have  $b(x) \neq x$ , for all  $x \in Fix(a)$ . Note also that, since  $G_1$  is rank-one Abelian group,  $Fix(a) = Fix(G_1)$ .

Now let  $x_{-1} = \inf\{t(g) \mid g \in G_1\}$  and  $x_1 = \sup\{t(g) \mid g \in G_1\}$ . It is easy to see that both  $x_{-1}$  and  $x_1$  are fixed points of  $G_1$ . Moreover,  $x_{-1}$  (resp.  $x_1$ ) is the first fixed point of  $a$  on the left (resp. right) of  $t(id)$ . In particular,  $b((x_{-1}, x_1)) \cap (x_{-1}, x_1) = \emptyset$ , since otherwise there would be a fixed point inside  $(x_{-1}, x_1)$ . Taking the *reverse* ordering if necessary, we may assume  $b \succ id$ . In particular, we have that  $b(x_{-1}) \geq x_1$ .

We now claim that  $G_1$  is a convex subgroup. First note that, by the definition of the dynamical realization, for every  $g \in G$  we have  $t(g) = g(t(id))$ . Then, it follows that for every  $g \in G_1$ ,  $t(g) \in (x_{-1}, x_1)$ . Now let  $m, s$  in  $\mathbb{Z}$  and  $g \in G_1$  be such that  $id \prec b^m g \prec a^s$ . Then we have  $t(id) < b^m(t(g)) < t(a^s) < x_1$ . Since  $b(x_{-1}) \geq x_1$ , this easily yields  $m = 0$ , that is,  $b^m g = g \in G_1$ .

We have thus proved that  $G_1$  is a convex (normal) subgroup of  $G$ . Since the quotient  $G/G_1$  is isomorphic to  $\mathbb{Z}$ , an almost direct application of Theorem 1.0.1 shows that the ordering  $\preceq$  is Conradian. This concludes the proof of Theorem 3.1.4.  $\square$

**Remark 3.1.7.** It follows from Theorem 3.1.4 and Proposition 3.1.3 that no left-ordering is isolated in  $\mathcal{LO}(G)$ . Therefore, since any group with normal rational series is countable,  $\mathcal{LO}(G)$  is a totally disconnected Hausdorff and compact metric space, thus homeomorphic to the Cantor set.

**Remark 3.1.8.** The preceding method of proof also gives a complete classification –up to topological semiconjugacy– of all actions of  $G$  by orientation-preserving homeomorphisms of the real line (compare [34]). In particular, all these actions come from left-orderings on the group (compare Question 2.4 in [31] and the comments before it). This has been recently used by Guelman and Liousse to classify all  $C^1$  actions of the solvable Baumslag-Solitar groups on the circle [17].

## 3.2 The general case

### 3.2.1 A technical proposition

The main purpose of this section is to prove the following

**Proposition 3.2.1.** *Let  $G$  be a group with only finitely many  $\mathcal{C}$ -orderings, and let  $H$  be its maximal proper convex subgroup (with respect to any  $\mathcal{C}$ -ordering). Then  $H$  is a Tararin group, that is, a group with only finitely many left-orderings.*

Note that the existence of a maximal convex subgroup follows from Theorem B. Note also that Proposition 3.2.1 implies that no group with only finitely many  $\mathcal{C}$ -orderings, whose rational series has length at least 3, is bi-orderable (see also [36, Proposition 3.2]).

The proof of Proposition 3.2.1 is a direct consequence of the following

**Lemma 3.2.2.** *Let  $G$  be a group with only finitely many  $\mathcal{C}$ -orderings whose rational series has length at least three:*

$$\{id\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_n = G, \quad n \geq 3. \quad (3.2)$$

*Then, given  $a \in G_1$  and  $b \in G_i$ ,  $i \leq n-1$ , we have that  $bab^{-1} = a^\varepsilon$ , where  $\varepsilon = \pm 1$ .*

*Proof:* We shall proceed by induction on  $i$ . For  $i = 0, 1$ , the conclusion is obvious. Let us deal with the case  $i = 2$ . Let  $b \in G_2$ , and suppose that  $bab^{-1} = a^r$ , where  $r \neq \pm 1$  is rational. Clearly, this implies that  $b^n ab^{-n} = a^{r^n}$  for all  $n \in \mathbb{Z}$ .

Since  $G_3/G_1$  is non Abelian, there exists  $c \in G_3$  such that  $cb^p c^{-1} = b^q w$ , with  $p \neq q$  integers and  $w \in G_1$ . Note that  $wa = aw$ . We let  $t \in \mathbb{Q}$  be such that  $cac^{-1} = a^t$ . Then we have

$$a^{r^q} = b^q ab^{-q} = b^q waw^{-1}b^{-q} = cb^p c^{-1}a cb^{-p} c^{-1} = cb^p a^{1/t} b^{-p} c^{-1} = ca^{\frac{r^p}{t}} c^{-1} = a^{r^p},$$

which is impossible since  $r \neq \pm 1$  and  $p \neq q$ . Thus the case  $i = 2$  is settled.

Now assume, as the induction hypothesis, that for any  $w \in G_{i-1}$  we have that  $waw^{-1} = a^\varepsilon$ ,  $\varepsilon = \pm 1$ . Suppose also that there exists  $b \in G_i$  such that  $bab^{-1} = a^r$ ,  $r \neq \pm 1$ . As before, we have that  $b^n ab^{-n} = a^{r^n}$  for all  $n \in \mathbb{Z}$ .

Let  $c \in G_{i+1}$  such that  $cb^p c^{-1} = b^q w$ , with  $p \neq q$  integers and  $w \in G_{i-1}$ . Let  $t \in \mathbb{Q}$  be such that  $cac^{-1} = a^t$ . Then we have

$$a^{r^q} = b^q ab^{-q} = b^q w w^{-1} a w w^{-1} b^{-q} = cb^p c^{-1} a^\varepsilon cb^{-p} c^{-1} = cb^p a^{\varepsilon/t} b^{-p} c^{-1} = ca^{\frac{\varepsilon r^p}{t}} c^{-1} = a^{\varepsilon r^p},$$

which is impossible since  $r \neq \pm 1$  and  $p \neq q$  imply  $|r^p| \neq |r^q|$ . This finishes the proof of Lemma 3.2.2.  $\square$

*Proof of Proposition 3.2.1:* Since in any Conradian ordering of  $G$ , the series of convex subgroups is precisely the (unique) rational series associated to  $G$ , we have that  $H = G_{n-1}$  in (3.2). So  $H$  has a rational normal series. Therefore, to prove that  $H$  is a Tararin group, we only need to check that no quotient  $G_i/G_{i-2}$ ,  $2 \leq i \leq n-1$ , is bi-orderable.

Now, if in (3.2) we take the quotient by the normal and convex subgroup  $G_{i-2}$ , Lemma 3.2.2 implies that certain element in  $G_{i-1}/G_{i-2}$  is sent into its inverse by the action of some element in  $G_i/G_{i-2}$ . Thus  $G_i/G_{i-2}$  is non bi-orderable.  $\square$

**Corollary 3.2.3.** *A group  $G$  having only finitely many  $\mathcal{C}$ -orderings, with rational series*

$$\{id\} \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G,$$

*is a Tararin group if and only if  $G/G_{n-2}$  is a Tararin group.*

### 3.2.2 The inductive step

Let  $G$  be a group with rational series

$$\{id\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G, \quad n \geq 3,$$

such that no quotient  $G_i/G_{i-2}$  is Abelian. Moreover, assume  $G$  is not a Tararin group. Let  $\preceq$  be a left-ordering on  $G$ . To show that  $\preceq$  is non-isolated we will proceed by induction. Therefore, we assume as induction hypothesis that no group with only finitely many  $\mathcal{C}$ -orderings, but infinitely many left-orderings, whose rational series has length less than  $n$ , has isolated left-orderings.

The main idea of the proof is to find a convex subgroup  $H$  such that either  $H$  has no isolated left-orderings or such that  $H$  is normal and  $G/H$  has no isolated left-orderings. We will see that the appropriate convex subgroup to look at is the *convex closure of  $G_1$*  (with respect to  $\preceq$ ), that is, the smallest convex subgroup that contains  $G_1$ .

For  $x, y$  in  $G$ , consider the relation in  $G$  given by  $x \sim y$  if and only if there are  $g_1, g_2$  in  $G_1$  such that  $g_1x \preceq y \preceq g_2x$ . We check that  $\sim$  is an equivalence relation. Clearly  $x \sim x$  for all  $x \in G$ . If  $x \sim y$  and  $y \sim z$ , then there are  $g_1, g_2, g'_1, g'_2$  in  $G_1$  such that  $g_1x \preceq y \preceq g_2x$  and  $g'_1y \preceq z \preceq g'_2y$ . Then  $g'_1g_1x \preceq z \preceq g'_2g_2x$ , hence  $x \sim z$ . Finally,  $g_1x \preceq y \preceq g_2x$  implies  $g_2^{-1}y \preceq x \preceq g_1^{-1}y$ , thus  $x \sim y \Rightarrow y \sim x$ .

Now let  $g, x, y$  in  $G$  be such that  $x \sim y$ . Then  $g_1x \preceq y \preceq g_2x$ , for some  $g_1, g_2$  in  $G_1$ , hence  $gg_1x \preceq gy \preceq gg_2x$ . Since  $G_1$  is normal, we have that  $gg_1x = g'_1gx$  and  $gg_2x = g'_2gx$ , for some  $g'_1, g'_2$  in  $G_1$ . Therefore,  $g'_1gx \preceq gy \preceq g'_2gx$ , so  $gx \sim gy$ . Thus,  $G$  preserves the equivalence relation  $\sim$ . Let  $H = \{x \in G \mid x \sim id\}$ .

Claim 1. For every  $g \in G$ , we have

$$gH \cap H = \begin{cases} \emptyset & \text{if } g \notin H, \\ H & \text{if } g \in H. \end{cases}$$

Indeed, if  $g \in H$ , then  $g \in (gH \cap H)$ . Now, since  $x \sim y \Leftrightarrow gx \sim gy$ , we have that  $gH = H$ . Now suppose  $g$  is such that there is some  $z \in gH \cap H$ . Then  $id \sim z \sim g$ , which implies  $g \in H$ . Therefore, Claim 1 follows.

Claim 1 implies that  $H$  is a convex subgroup of  $G$  that contains  $G_1$ . Moreover, we have

Claim 2. The subgroup  $H$  is the convex closure of the subgroup  $G_1$ .

Indeed, let  $C$  denote the convex closure of  $G_1$  in  $\preceq$ . Then  $H$  is a convex subgroup that contains  $G_1$ . Thus  $C \subseteq H$ .

To show that  $H \subseteq C$  we just note that, by definition, for every  $h \in H$ , there are  $g_1, g_2$  in  $G_1$  such that  $g_1 \preceq h \preceq g_2$ . So  $H \subseteq C$ , and Claim 2 follows.

Proceeding as in Lemma 3.1.1, we conclude that there is  $c \in G$  such that  $cG_{n-1}$  generates the quotient  $G/G_{n-1}$ . We have

Claim 3.  $H/G_1$  is either trivial or isomorphic to  $\mathbb{Z}$ .

By proposition 3.2.1  $G_{n-1}$  is a Tararin group. Therefore, in the restriction of  $\preceq$  to  $G_{n-1}$ ,  $G_1$  is convex. So we have that  $H \cap G_{n-1} = G_1$ . This means that for every  $g \in G_{n-1} \setminus G_1$ , one has  $gH \cap H = \emptyset$ .

Now, assume  $H/G_1$  is nontrivial and let  $g \in H \setminus G_1$ . By the preceding paragraph, we have that  $g \notin G_{n-1}$ . Therefore,  $g = c^{m_1}w_{m_1}$ , for  $m_1 \in \mathbb{Z}$ ,  $m_1 \neq 0$  and  $w_{m_1} \in G_{n-1}$ .

Let  $m_0$  be the least positive  $m \in \mathbb{Z}$  such that  $c^m w_m \in H$ , for  $w_m \in G_{n-1}$ . Then, by the minimality of  $m_0$ , we have that  $m_1$  is a multiple of  $m_0$ , say  $km_0 = m_1$ . Letting  $(c^{m_0} w_{m_0})^k = c^{m_0 k} \overline{w_{m_0}}$ , we have that  $(c^{m_0} w_{m_0})^{-k} c^m w_m = \overline{w_{m_0}}^{-1} w_m \in H$ . Since  $\overline{w_{m_0}}^{-1} w_m \in G_{n-1}$ , we have that  $\overline{w_{m_0}}^{-1} w_m \in G_1$ . Therefore we conclude that  $(c_0^m w_{m_0})^k G_1 = c^m w_m G_1$ , which proves our Claim 3.

We are now in position to finish the proof of the Theorem D. According to Claim 3 above, we need to consider two cases.

**Case 1.**  $H = G_1$ .

In this case,  $G_1$  is a convex normal subgroup of  $\preceq$  and, since by induction hypothesis  $G/G_1$  has no isolated left-orderings,  $\preceq$  is non-isolated.

**Case 2.**  $H/G_1 \simeq \mathbb{Z}$ .

In this case,  $H$  has a rational series of length 2:

$$\{id\} = G_0 \triangleleft G_1 \triangleleft H.$$

We let  $a \in G_1$ ,  $a \neq id$ , and  $h \in H$  be such that  $hG_1$  generates  $H/G_1$ . Let  $r \in \mathbb{Q}$  be such that  $hah^{-1} = a^r$ . We have three subcases:

*Subcase 1.*  $r < 0$ .

Clearly, in this subcase,  $H$  is non bi-orderable. Thus  $H$  is a Tararin group and  $G_1$  is convex in  $H$ . However, as proved in Claim 2,  $H$  is the convex closure of  $G_1$ . Therefore, this subcase does not arise.

*Subcase 2.*  $r > 0$ .

Since  $r > 0$ , we have that  $H$  is not a Tararin group, thus  $H$  has no isolated left-orderings. Therefore,  $\preceq$  is non-isolated.

*Subcase 3.*  $r = 0$ .

In this case,  $H$  is a rank-two Abelian group, so it has no isolated orderings. Hence  $\preceq$  is non-isolated.

This finishes the proof of Theorem D.

### 3.2.3 An illustrative example

This subsection is aimed to illustrate the different kind of left-orderings that may appear in a group as above. To do this, we will consider a family of groups with eight  $\mathcal{C}$ -orderings. We let  $G(n) = \langle a, b, c \mid bab^{-1} = a^{-1}, cbc^{-1} = b^3, cac^{-1} = a^n \rangle$ , where  $n \in \mathbb{Z}$ . It is easy to see that  $G(n)$  has a rational series of length three:

$$\{id\} \triangleleft G_1 = \langle a \rangle \triangleleft G_2 = \langle a, b \rangle \triangleleft G(n).$$

In particular, in a Conradian ordering,  $G_1$  is convex and normal.

Now we note that  $G(n)/G_1 \simeq B(1, 3)$ , where  $B(1, 3) = \langle \beta, \gamma \mid \gamma\beta\gamma^{-1} = \beta^3 \rangle$  is a Baumslag-Solitar group, where the isomorphism is given by  $c \mapsto \gamma$ ,  $b \mapsto \beta$ ,  $a \mapsto id$ . Now consider the (faithful) representation  $\varphi : B(1, 3) \rightarrow Homeo_+(\mathbb{R})$  of  $B(1, 3) \simeq G(n)/G_1$  into  $Homeo_+(\mathbb{R})$  given by  $\varphi(\beta)(x) = x + 1$  and  $\varphi(\gamma)(x) = 3x$ . It is easy to see that if  $x \in \mathbb{R}$ , then  $Stab_{\varphi(B(1, 3))}(x)$  is either trivial or isomorphic to  $\mathbb{Z}$ .

In particular,  $\text{Stab}_{\varphi(B(1,3))}(\frac{-3k}{2}) = \langle \gamma\beta^k \rangle$ , where  $k \in \mathbb{Z}$ . Thus  $\langle \gamma\beta^k \rangle$  is convex in the induced ordering from the point  $\frac{-3k}{2}$  (in the representation given by  $\varphi$ ). Now, using the isomorphism  $G(n)/G_1 \simeq B(1,3)$ , we have induced an ordering on  $G(n)/G_1$  with the property that  $\langle cb^k G_1 \rangle$  is convex. We denote this left-ordering by  $\preceq_2$ . Now, extending  $\preceq_2$  by the initial Conradian ordering on  $G_1$ , we have created an ordering  $\preceq$  on  $G(n)$  with the property that  $H(n) = \langle a, cb^k \rangle$  is convex. Moreover, we have:

- If  $n = 1$  and  $k = 0$ , then  $H(n) = \langle a, c \rangle \leq G(n)$  is convex in  $\preceq$  and  $ca = ac$ , as in Subcase 3 above.
- If  $n \geq 2$ , and  $k = 0$ , then  $H(n) = \langle a, c \rangle \leq G(n)$  is convex in  $\preceq$  and  $cac^{-1} = a^2$ , as in Subcase 2 above.
- If  $n \leq -1$  and  $k$  is odd, then  $H(n) = \langle a, cb^k \rangle \leq G(n)$  is convex and  $cb^k a b^{-k} c^{-1} = a^{-n}$ , (again) as in Subcase 2 above.

## Chapter 4

# On the space of left-orderings of the free group

As announced in the Introduction, in this chapter we give an explicit construction leading to a proof of the following theorem obtained by Clay in [8]:

**Theorem F (Clay).** *The space of left-orderings of the free group on two or more generators  $F_n$  has a dense orbit under the natural conjugacy action of  $F_n$ .*

We must note that, though stated for  $F_n$ ,  $n \geq 2$ , we will only deal with the case  $n = 2$ . Nevertheless, our method extends in a rather obvious way for the general case (see Remark 4.1.7).

As a Corollary, and following [8], we next re-prove McCleary's theorem from [24] asserting that the space of left-orderings of the free group on two or more generators has no isolated points, hence it is homeomorphic to a Cantor set. Indeed, we have the following general

**Proposition 4.0.4.** *Suppose  $G$  is an infinite, left-orderable group such that  $\mathcal{LO}(G)$  contains a dense orbit for the conjugacy action of  $G$  on  $\mathcal{LO}(G)$ . Then  $\mathcal{LO}(G)$  contains no isolated points.*

*Proof:* Let  $\preceq_D$  be an ordering with dense orbit in  $\mathcal{LO}(G)$ . We distinguish two cases.

**Case 1.**  $\preceq_D$  is non isolated.

In this case, since the action of  $G$  on  $\mathcal{LO}(G)$  is by homeomorphism, we have that no point in the orbit of  $\preceq_D$  is isolated. In particular, no point in  $\mathcal{LO}(G)$  is isolated.

**Case 2.**  $\preceq_D$  is isolated.

If  $\preceq_D$  is isolated, then its reverse ordering  $\overline{\preceq}_D$  is also isolated (recall that, for any  $f \in G$ ,  $id \prec_D f$  if and only if  $id \succ_D f$ ). This implies that  $\preceq_D \in \text{Orb}_G(\preceq_D)$ . Hence, there exists  $g \in G$  such that  $g(\preceq_D) = \overline{\preceq}_D$ . By the definition of the action, this means that  $gfg^{-1} \prec_D id$ , for any  $f \succ_D id$ . But this is impossible, since the  $\preceq_D$ -signs of  $g$  and  $g^{-1}$  are preserved under conjugation by  $g$ . Thus Case 2 never arises.  $\square$

### 4.1 Constructing a dense orbit

Let  $\preceq$  be a left-ordering on  $F_2 = \langle a, b \rangle$ . Let  $D : F_2 \rightarrow \text{Homeo}_+(\mathbb{R})$  be an homomorphic embedding with the property that *there exists  $x \in \mathbb{R}$  such that  $g \succ id$  if and only if  $D(g)(x) > x$* . We call  $D$  a *dynamical realization-like homomorphism* for  $\preceq$ . The point  $x$  is called *reference point* for  $D$ .

**Example 4.1.1.** The embedding given by any dynamical realization of any countable left-ordered group  $(G, \preceq)$  is a dynamical realization-like homomorphism for  $\preceq$  with reference point  $t_{\preceq}(id)$ .

**Definition 4.1.2.** Let  $B_n = \{w \in F_2 = \langle a, b \rangle \mid |w| \leq n\}$ , where  $|w|$  represents the length of the element  $w$ , be the *ball* of radius  $n$  in  $F_2$ . Given  $B_n \subseteq F_2$  and a left-ordering  $\preceq$  of  $F_2$ , let

$$g_{(B_n, \preceq)}^- = \min_{\preceq} \{w \in B_n\}, \quad g_{(B_n, \preceq)}^+ = \max_{\preceq} \{w \in B_n\}.$$

Now let  $D$  be a dynamical realization-like homomorphism for  $\preceq$ , with reference point  $x$ . Then, we will refer to the square  $[D(g_{(B_n, \preceq)}^-)(x), D(g_{(B_n, \preceq)}^+)(x)]^2 \subset \mathbb{R}^2$  as the  $(B_n, \preceq)$ -box.

We now proceed to the construction of a nice action of  $F_2$  on  $\mathbb{R}$ . Let  $\mathcal{D} = \{\preceq_1, \preceq_2, \dots\}$  be a countable dense subset of  $\mathcal{LO}(F_2)$ . Let  $\mathcal{B} = \{B_n\}_{n=1}^\infty$  be the (countable) set of all balls in  $F_2$ . Let  $\eta : \mathbb{Z} \rightarrow \mathcal{B} \times \mathcal{D}$  be a surjection, with  $\eta(k) = (B_{n_k}, \preceq_{m_k})$ .

Note that, if  $D$  is any dynamical realization-like homomorphism for  $\preceq$ , with reference point  $x$ , and if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is any increasing continuous function, then the conjugated homomorphism  $D_\varphi$  defined by  $D_\varphi(g) = \varphi D(g) \varphi^{-1}$  is again a dynamical realization-like homomorphism for  $\preceq$  but with reference point  $\varphi(x)$ . Therefore, for  $\eta(k) = (B_{n_k}, \preceq_{m_k})$ , we may let  $D_{\eta(k)} : F_2 \rightarrow \text{Homeo}_+(\mathbb{R})$  be a dynamical realization-like homomorphism for  $\preceq_{m_k}$  such that:

- (i) The reference point for  $D_{\eta(k)}$  is  $k$ .
- (ii) The  $\eta(k)$ -box coincides with the square  $[k - 1/3, k + 1/3]^2$ .

The next lemma shows that, in the action given by  $D_{\eta(k)}$ , the  $\preceq_{m_k}$ -signs of elements in  $B_{n_k}$  are contained as part of the information of the *graphs*<sup>1</sup> of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$  inside the  $\eta(k)$ -box.

**Lemma 4.1.3.** *Let  $\eta(k) = (B_{n_k}, \preceq_{m_k})$ , and let  $D_{\eta(k)}$  be dynamical realization-like homomorphisms satisfying properties (i) and (ii) above. Then, for every  $w \in B_{n_k}$ , we have that  $D_{\eta(k)}(w)(k)$  belongs to  $[k - 1/3, k + 1/3]$ , and  $D_{\eta(k)}(w) > k$  if and only if  $w \succ_{m_k} \text{id}$ . Moreover, we have  $D_{\eta(k)}(g_{\eta(k)}^+)(k) = k + 1/3$  and  $D_{\eta(k)}(g_{\eta(k)}^-)(k) = k - 1/3$ .*

*Proof:* Since  $D_{\eta(k)}$  is a dynamical realization-like homomorphism, property (i) above implies that, for any  $w \in F_2$ , we have that  $D_{\eta(k)}(w)(k) > k$  if and only if  $w \succ_{m_k} \text{id}$ .

The fact that  $D_{\eta(k)}(g_{\eta(k)}^+)(k) = k + 1/3$  and  $D_{\eta(k)}(g_{\eta(k)}^-)(k) = k - 1/3$  is a direct consequence of property (ii) above.

Finally, note that for every  $w \in B_{n_k}$  we have  $g_{\eta(k)}^- \preceq_{m_k} w \preceq_{m_k} g_{\eta(k)}^+$ . In particular,  $D_{\eta(k)}(w)(k) \in [k - 1/3, k + 1/3]$ .  $\square$

**Remark 4.1.4.** Note that every initial segment  $w_1$  of any (reduced) word  $w \in B_{n_k}$  lies again in  $B_{n_k}$ . Hence, the iterates of  $k$  along the word  $w$  are independent of the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$  outside the  $\eta(k)$ -box  $= [k - 1/3, k + 1/3]^2$ . In particular, for any representation  $D : F_2 \rightarrow \text{Homeo}_+(\mathbb{R})$  such that the graphs of  $D(a)$  and  $D(b)$  coincide with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$  inside  $[k - 1/3, k + 1/3]^2$ , respectively, the conclusion of Lemma 4.1.3 holds for  $D$  instead of  $D_{\eta(k)}$ .

Theorem F is a direct consequence of the following

**Proposition 4.1.5.** *Let  $F_2 = \langle a, b \rangle$ . Then there is an homomorphic embedding  $D : F_2 \rightarrow \text{Homeo}_+(\mathbb{R})$  such that, for each  $k \in \mathbb{Z}$ , the graphs of  $D(a)$  and  $D(b)$  inside  $[k - 1/3, k + 1/3]^2$  coincide with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$ , respectively. In this action, all the integers lie in the same orbit.*

<sup>1</sup>As usual, for  $f \in \text{Homeo}_+(\mathbb{R})$ , the set  $\{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$  is called the graph of  $f$ .

*Proof of Theorem F from Proposition 4.1.5:* Let  $(x_0, x_1, \dots)$  be a dense sequence in  $\mathbb{R}$  such that  $x_0 = 0$  (note that 0 may not have a free orbit), and let  $D$  be the homomorphic embedding given by Proposition 4.1.5. Let  $\preceq$  be the induced ordering on  $F_2 = \langle a, b \rangle$  from the action  $D$  and the reference points  $(x_0, x_1, x_2, \dots)$  (see the comments after Theorem 1.4.1). In particular, for  $g \in F_2$ , we have that  $D(g)(0) > 0 \Rightarrow g \succ id$ . We claim that  $\preceq$  has a dense orbit under the natural action of  $F_2$  on  $\mathcal{LO}(F_2)$ .

Clearly, to prove our claim it is enough to prove that the orbit of  $\preceq$  accumulates at every  $\preceq_m \in \mathcal{D}$ . That is, given  $\preceq_m$  and any finite set  $\{h_1, h_2, \dots, h_N\}$  such that  $id \prec_m h_j$ , for  $1 \leq j \leq N$ , we need to find  $w \in F_2$  such that  $h_j \succ_w id$  for every  $1 \leq j \leq N$ , where, as defined in §1.3.1,  $g \succ_w id$  if and only if  $wgw^{-1} \succ id$ .

Let  $n \in \mathbb{N}$  be such that  $h_1, \dots, h_N$  belongs to  $B_n$ . Let  $k$  be such that  $\eta(k) = (B_n, \preceq_m)$ . By Proposition 4.1.5, there is  $w_k \in F_2$  such that  $D(w_k)(0) = k$ . Also by Proposition 4.1.5, the graphs of  $D(a)$  and  $D(b)$ , inside  $[k - 1/3, k + 1/3]^2 = \eta(k)$ -box, are the same as those of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$ , respectively. Then, Lemma 4.1.3 implies that for each  $h_j$ ,  $1 \leq j \leq N$ , we have that  $h_j \succ_m id$  if and only if  $D(h_j)(k) > k$ . But this is the same as saying that  $D(h_j)(D(w_k)(0)) > D(w_k)(0)$ , which implies that  $D(w_k^{-1}) \circ D(h_j) \circ D(w_k)(0) > 0$ , where  $\circ$  is the composition operation. Therefore, by definition of  $\preceq$ , we have that  $w_k^{-1} h_j w_k \succ id$  for every  $1 \leq j \leq N$ . Now, by definition of the action of  $F_2$  on  $\mathcal{LO}(F_2)$ , this implies that  $\preceq_{w_k^{-1}}$  is a left-ordering such that  $h_j \succ_{w_k^{-1}} id$ . This finishes the proof of Theorem F.  $\square$

Before proving Proposition 4.1.5 we need one more lemma. Let  $\hat{a}$  and  $\hat{b}$  be two increasing continuous functions of the real line such that, for each  $k \in \mathbb{Z}$ , the graphs of  $\hat{a}$  and  $\hat{b}$  inside  $[k - 1/3, k + 1/3]^2$  coincide with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$ , respectively. We have

**Lemma 4.1.6.** *For each  $k \in \mathbb{Z}$ , we can modify the homeomorphisms  $\hat{a}$  and  $\hat{b}$  inside  $[k - 1/3, k + 1 + 1/3]^2$  but outside  $[k - 1/3, k + 1/3]^2 \cup [k + 1 - 1/3, k + 1 + 1/3]^2$  (see Figure 4.1) in such a way that the modified homeomorphisms, which we still denote  $\hat{a}$  and  $\hat{b}$ , have the following property P:*

*There is a reduced word  $w$  in the free group generated by  $\{\hat{a}, \hat{b}\}$  such that  $w(k) = k + 1$ . Moreover, the iterates of  $k$  along  $w$  remain inside  $[k - 1/3, k + 1 + 1/3]$ .*

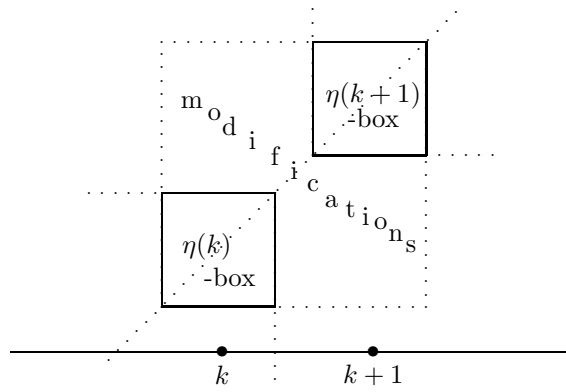


Figure 4.1

*Proof:* Note that, since for each  $k \in \mathbb{Z}$  the graphs of  $\hat{a}$  and  $\hat{b}$  coincide with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$ , respectively, by Remark 4.1.4 we have that  $k, k + 1/3$  and  $k - 1/3$  are in the same



orbit (for the action of the free group generated by  $\hat{a}$  and  $\hat{b}$ ). Therefore, to show this lemma, it is enough to show that we can modify  $\hat{a}$  and  $\hat{b}$  in such a way that the following property  $P'$  holds:

*There is a reduced word  $w$  in the free group generated by  $\{\hat{a}, \hat{b}\}$  such that  $w(k+1/3) = k+1-1/3$ . Moreover, the iterates of  $k+1/3$  along  $w$  remain inside  $[k-1/3, k+1+1/3]$ .*

For  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$  define  $l_h = \sup\{x \in [k-1/3, k+1/3] \mid h(x) \leq k+1/3\}$  and  $r_h = \inf\{x \in [k+1-1/3, k+1+1/3] \mid h(x) \geq k+1-1/3\}$ . Let  $x_0 \in ]k+1/3, k+1-1/3[$ . To modify  $\hat{a}$  and  $\hat{b}$ , we proceed as follows:

**Case 1:** There is  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$  such that  $l_h < k+1/3$  and  $r_h = k+1-1/3$ .

In this case, we (re)define  $h$  linearly from  $(l_h, h(l_h)) = (l_h, k+1/3)$  to  $(k+1/3, x_0)$ , then linearly from  $(k+1/3, x_0)$  to  $(x_0, k+1-1/3)$ , and then linearly from  $(x_0, k+1-1/3)$  to  $(k+1-1/3, h(k+1-1/3)) = (r_h, h(r_h))$ ; see Figure 4.2 (a). The other generator, say  $f$ , may be extended linearly from  $(l_f, f(l_f))$  to  $(r_f, f(r_f))$ .

Note that in this case we have  $h(k+1/3) = x_0$  and  $h(x_0) = k+1-1/3$ . This shows that  $P'$  holds for  $w = h^2$ .

We note that, for  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$ , we have that  $l_h = k+1/3 \Leftrightarrow l_{h^{-1}} < k+1/3$  and  $r_h = k+1-1/3 \Leftrightarrow r_{h^{-1}} > k+1-1/3$ . Therefore, if there is no  $h$  as in Case 1, then we are in

**Case 2:** There are  $f, h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$  such that  $l_h < k+1/3$ ,  $r_h > k+1-1/3$ ,  $l_f < k+1/3$  and  $r_f > k+1-1/3$ .

In this case we define  $h$  linearly from  $(l_h, h(l_h))$  to  $(k+1/3, x_0)$ , and then linearly from  $(k+1/3, x_0)$  to  $(r_h, h(r_h))$ . For  $f$ , we define it linearly from  $(l_f, f(l_f))$  to  $(k+1-1/3, x_0)$ , and then linearly from  $(k+1-1/3, x_0)$  to  $(r_f, f(r_f))$ ; see Figure 4.2 (b).

Note that  $h(k+1/3) = x_0 = f(k+1-1/3)$ . This shows that  $P'$  holds for  $w = f^{-1}h$ .  $\square$

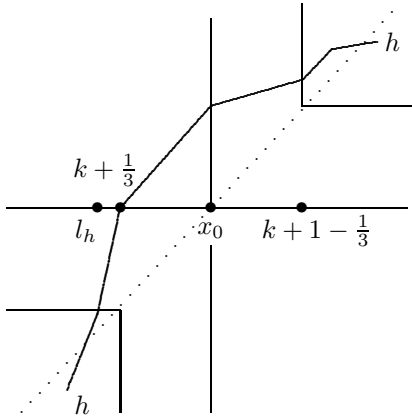


Figure 4.2 (a)

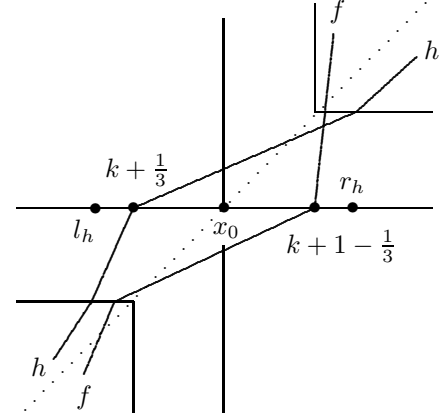


Figure 4.2 (b)

*Proof of Proposition 4.1.5:* We let  $\hat{a}$  and  $\hat{b}$  be as in Lemma 4.1.6. For each  $k \in \mathbb{Z}$  we apply inductively Lemma 4.1.6 to modify  $\hat{a}$  and  $\hat{b}$  inside  $[k-1/3, k+1+1/3]^2$ . Note that property  $P$  implies that these modifications are made in such a way that they do not overlap one with each other. In particular, for each  $k \in \mathbb{Z}$ , the graphs of  $\hat{a}$  and  $\hat{b}$  coincides with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$  inside  $[k-1/3, k+1/3]^2$ . Moreover, since  $k$  and  $k+1$  lie in the same orbit for all  $k$ , we have that all the integers are in the same orbit.

We thus let  $D : F_2 \rightarrow \text{Homeo}_+(\mathbb{R})$  be the homomorphism defined by  $D(a) = \hat{a}$  and  $D(b) = \hat{b}$ . To see that  $D$  is an embedding we just have to note that any  $w \in B_{n_k}$ , where  $\eta(k) = (B_{n_k}, \preceq_{m_k})$ , acts nontrivially at the point  $k \in \mathbb{R}$ . Indeed, since  $D(w)(k) = D_{\eta(k)}(w)(k)$ , Lemma 4.1.3 applies. This finishes the proof of Proposition 4.1.5  $\square$

**Remark 4.1.7.** We point out that only two technical facts were needed in the proof above. The first one is that the partial dynamics in the  $(B_n, \preceq_m)$ -box contains the information of the  $\preceq_m$ -sings of the elements of  $B_n$ . The second is that we can glue all these boxes together in a sole action of  $F_2$  so that there is a single orbit containing all the centers of the boxes.

In the case of a general free group  $F_n = \langle a_1, \dots, a_n \rangle$ ,  $n \geq 2$ , the first of these facts clearly holds. The second fact can be ensured by performing the same construction taking  $a = a_1$ , and  $b = a_2$ , whereas the remaining generators are extended linearly from the edge of one box to the edge of the following box. This gives an action of  $F_n$  for which there is a single orbit containing the centers of the boxes, which allows conclude as in the case of  $F_2$ .

## Chapter 5

# Describing all bi-orderings on Thompson's group $F$

In this chapter, we focus on a remarkable bi-orderable group, namely Thompson's group  $F$ , and we provide a complete description of all its possible bi-orderings. This is essentially taken from [32].

Recall that  $F$  is the group of orientation-preserving piecewise-linear homeomorphisms  $f$  of the interval  $[0, 1]$  such that:

- the derivative of  $f$  on each linearity interval is an integer power of 2,
- $f$  induces a bijection of the set of dyadic rational numbers in  $[0, 1]$ .

For each nontrivial  $f \in F$  we will denote by  $x_f^-$  (resp.  $x_f^+$ ) the leftmost point  $x^-$  (resp. the rightmost point  $x^+$ ) for which  $f'_+(x^-) \neq 1$  (resp.  $f'_-(x^+) \neq 1$ ), where  $f'_+$  and  $f'_-$  stand for the corresponding lateral derivatives. One can then immediately visualize four different bi-orderings on (each subgroup of)  $F$ , namely:

- the bi-ordering  $\preceq_{x^-}^+$  for which  $f \succ id$  if and only if  $f'_+(x_f^-) > 1$ ,
- the bi-ordering  $\preceq_{x^-}^-$  for which  $f \succ id$  if and only if  $f'_+(x_f^-) < 1$ ,
- the bi-ordering  $\preceq_{x^+}^+$  for which  $f \succ id$  if and only if  $f'_-(x_f^+) < 1$ ,
- the bi-ordering  $\preceq_{x^+}^-$  for which  $f \succ id$  if and only if  $f'_-(x_f^+) > 1$ .

Although  $F$  admits many more bi-orderings than these, the case of its derived subgroup  $F'$  is quite different. As discussed in the Introduction, this particular case is related to Dlab's work [12]. In §5.1 we show

**Theorem 5.0.8.** *The only bi-orderings on  $F'$  are  $\preceq_{x^-}^+$ ,  $\preceq_{x^-}^-$ ,  $\preceq_{x^+}^+$  and  $\preceq_{x^+}^-$ .*

Remark that there are also four other “exotic” bi-orderings on  $F$ , namely:

- the bi-ordering  $\preceq_{0,x^-}^{+,-}$  for which  $f \succ id$  if and only if either  $x_f^- = 0$  and  $f'_+(0) > 1$ , or  $x_f^- \neq 0$  and  $f'_+(x_f^-) < 1$ ,
- the bi-ordering  $\preceq_{0,x^-}^{-,+}$  for which  $f \succ id$  if and only if either  $x_f^- = 0$  and  $f'_+(0) < 1$ , or  $x_f^- \neq 0$  and  $f'_+(x_f^-) > 1$ ,
- the bi-ordering  $\preceq_{1,x^+}^{+,-}$  for which  $f \succ id$  if and only if either  $x_f^+ = 1$  and  $f'_-(1) < 1$ , or  $x_f^+ \neq 1$  and  $f'_-(x_f^+) > 1$ ,
- the bi-ordering  $\preceq_{1,x^+}^{-,+}$  for which  $f \succ id$  if and only if either  $x_f^+ = 1$  and  $f'_-(1) > 1$ , or  $x_f^+ \neq 1$  and  $f'_-(x_f^+) < 1$ .

Notice that, when restricted to  $F'$ , the bi-ordering  $\preceq_{0,x-}^{+,-}$  (resp.  $\preceq_{0,x-}^{-,+}$ ,  $\preceq_{1,x+}^{+,-}$ , and  $\preceq_{1,x+}^{-,+}$ ) coincides with  $\preceq_{x-}^{-}$  (resp.  $\preceq_{x-}^{+}$ ,  $\preceq_{x+}^{-}$ , and  $\preceq_{x+}^{+}$ ). Let us denote the set of the previous eight bi-orderings on  $F$  by  $\mathcal{BO}_{\text{Isol}}(F)$ .

**Remark 5.0.9.** As the reader can easily check, the bi-ordering  $\preceq_{0,x-}^{+,-}$  appears as the extension by  $\preceq_{x-}^{+}$  of the restriction of its reverse ordering  $\preceq_{x-}^{+}$  (which coincides with  $\preceq_{x-}^{-}$ ) to the maximal proper  $\preceq_{x-}^{+}$ -convex subgroup  $F^{\text{max}} = \{f \in F : f'_+(0) = 1\}$ . The bi-orderings  $\preceq_{0,x-}^{-,+}$ ,  $\preceq_{1,x+}^{+,-}$ , and  $\preceq_{1,x+}^{-,+}$  may be obtained in the same way starting from  $\preceq_{x-}^{-}$ ,  $\preceq_{x+}^{+}$ , and  $\preceq_{x+}^{-}$ , respectively.

There is another natural procedure for creating bi-orderings on  $F$ . For this, recall the well-known fact that  $F'$  coincides with the subgroup of  $F$  formed by the elements  $f$  satisfying  $f'_+(0) = f'_-(1) = 1$ . Now let  $\preceq_{\mathbb{Z}^2}$  be any bi-ordering on  $\mathbb{Z}^2$ , and let  $\preceq_{F'}$  be any bi-ordering on  $F'$ . It readily follows from Theorem 5.0.8, that  $\preceq_{F'}$  is invariant under conjugacy by elements in  $F$ . Hence, from Corollary 1.2.5, we may define a bi-ordering  $\preceq$  on  $F$  by declaring that  $f \succ id$  if and only if either  $f \notin F'$  and  $(\log_2(f'_+(0)), \log_2(f'_-(1))) \succ_{\mathbb{Z}^2} (0, 0)$ , or  $f \in F'$  and  $f \succ_{F'} id$ .

All possible ways of ordering finite-rank Abelian groups have been described in [37, 39] (see Example 1.3.1 for the description of the space of orderings of  $\mathbb{Z}^2$ ). In particular, when the rank is greater than one, the corresponding spaces of bi-orderings are homeomorphic to the Cantor set. Since there are only four possibilities for the bi-ordering  $\preceq_{F'}$ , the preceding procedure gives four natural copies (which we will coherently denote by  $\Lambda_{x-}^{+}$ ,  $\Lambda_{x-}^{-}$ ,  $\Lambda_{x+}^{+}$ , and  $\Lambda_{x+}^{-}$ ) of the Cantor set in the space of bi-orderings of  $F$ . The main result of this chapter establishes that these bi-orderings, together with the special eight bi-orderings previously introduced, fill out the list of all possible bi-orderings on  $F$ .

**Theorem G .** *The space of bi-orderings of  $F$  is the disjoint union of the finite set  $\mathcal{BO}_{\text{Isol}}(F)$  (whose elements are isolated bi-orderings) and the copies of the Cantor set  $\Lambda_{x-}^{+}$ ,  $\Lambda_{x-}^{-}$ ,  $\Lambda_{x+}^{+}$ , and  $\Lambda_{x+}^{-}$ .*

The first ingredient of the proof of this result comes from the theory of Conradian orderings. Indeed, since  $F$  is finitely generated, see [6], every bi-ordering  $\preceq$  on it admits a maximal proper convex subgroup  $F_{\preceq}^{\text{max}}$ . More importantly, this subgroup may be detected as the kernel of a nontrivial, non-decreasing group homomorphism into  $(\mathbb{R}, +)$ ; see Theorem 1.0.1. Since  $F'$  is simple (see for instance [6]) and non Abelian, it must be contained in  $F_{\preceq}^{\text{max}}$ . The case of coincidence is more or less transparent: the bi-ordering on  $F$  is contained in one of the four canonical copies of the Cantor set, and the corresponding bi-ordering on  $\mathbb{Z}^2$  is of *irrational type* (see Example 1.3.1). The case where  $F'$  is strictly contained in  $F_{\preceq}^{\text{max}}$  is more complicated. The bi-ordering may still be contained in one of the four canonical copies of the Cantor set, but the corresponding bi-ordering on  $\mathbb{Z}^2$  must be of *rational type* (e.g., a lexicographic ordering). However, it may also coincide with one of the eight special bi-orderings listed above. Distinguishing these two possibilities is the hardest part of the proof. For this, we strongly use the internal structure of  $F$ , in particular the fact that the subgroup consisting of elements whose support is contained in a prescribed closed dyadic interval is isomorphic to  $F$  itself.

**Remark 5.0.10.** In general, if  $\Gamma$  is a finitely generated (nontrivial) group endowed with a bi-ordering  $\preceq$ , one can easily check that the ordering  $\preceq^*$  obtained as the extension by  $\preceq$  of  $\preceq$  restricted to  $\Gamma_{\preceq}^{\text{max}}$  is bi-invariant. This bi-ordering (resp. its conjugate  $\preceq_*$ ) is always different from  $\preceq$  (resp. from  $\preceq^*$ ), and it coincides with  $\preceq$  (resp. with  $\preceq^*$ ) if and only if the only proper  $\preceq$ -convex subgroup is the trivial one; by Conrad's theorem,  $\Gamma$  is necessarily Abelian in this case. We thus conclude that every non Abelian finitely generated bi-orderable group admits at least four different bi-orderings.

Moreover, (nontrivial) torsion-free Abelian groups having only two bi-orderings are those of rank-one (in higher rank one may consider lexicographic type orderings).

From Section 1.3.1 we have that  $Out(F)$  could be useful for understanding  $\mathcal{BO}(F)$ . Nevertheless, in the case of Thompson's group  $F$ , the action of  $Out(F)$  on  $\mathcal{BO}(F)$  is almost trivial. Indeed, according to [3], the group  $Out(F)$  contains an index-two subgroup  $Out_+(F)$  whose elements are (equivalence classes of) conjugacies by certain orientation-preserving homeomorphisms of the interval  $[0, 1]$ . Although these homeomorphisms are dyadically piecewise-affine on  $]0, 1[$ , the points of discontinuity of their derivatives may accumulate at 0 and/or 1, but in some "periodically coherent" way. It turns out that the conjugacies by these homeomorphisms preserve the derivatives of nontrivial elements  $f \in F$  at the points  $x_f^-$  and  $x_f^+$ : this is obvious when these points are different from 0 and 1, and in the other case this follows from the explicit description of  $Out(F)$  given in [3]. So, according to Theorem G, this implies that the action of  $Out_+(F)$  on  $\mathcal{BO}(F)$  is trivial.

The set  $Out(F) \setminus Out_+(F)$  corresponds to the class of the order-two automorphism  $\sigma$  induced by the conjugacy by the map  $x \mapsto 1 - x$ . One can easily check that

$$(\preceq_{x^-}^+)_{\sigma} = \preceq_{x^+}^-, \quad (\preceq_{x^-}^-)_{\sigma} = \preceq_{x^+}^+, \quad (\preceq_{0,x^-}^{+, -})_{\sigma} = \preceq_{1,x^+}^{-, +}, \quad \text{and} \quad (\preceq_{0,x^-}^{-, +})_{\sigma} = \preceq_{1,x^+}^{+, -}. \quad (5.1)$$

Moreover,  $\sigma(\Lambda_{x^-}^+) = \Lambda_{x^+}^-$  and  $\sigma(\Lambda_{x^-}^-) = \Lambda_{x^+}^+$ , and the action on the bi-orderings of the  $\mathbb{Z}^2$ -fiber can be easily described. We leave the details to the reader.

## 5.1 Bi-orderings on $F'$

For each dyadic (open, half-open, or closed) interval  $I$ , we will denote by  $F_I$  the subgroup of  $F$  formed by the elements whose *support*<sup>1</sup> is contained in  $I$ . Notice that if  $I$  is closed, then  $F_I$  is isomorphic to  $F$ . Therefore, for every closed dyadic interval  $I \subset ]0, 1[$ , every bi-ordering  $\preceq^*$  on  $F'$  gives rise to a bi-ordering on  $F \sim F_I$ . Moreover, if we fix such an  $I$ , then the induced bi-ordering on  $F_I$  completely determines  $\preceq^*$  (this is due to the invariance by conjugacy). The content of Theorem 5.0.8 consists of the assertion that only a few (namely four) bi-orderings on  $F_I$  may be extended to bi-orderings on  $F'$ . To prove this result, we will first focus on a general property of bi-orderings on  $F$ .

Let  $\preceq$  be a bi-ordering on  $F$ . Since bi-invariant orderings are Conradian and  $F$  is finitely generated, Theorem 1.0.1 provides us with a (unique up to a positive scalar factor) non-decreasing group homomorphism  $\tau_{\preceq}: F \rightarrow (\mathbb{R}, +)$ , called the *Conrad homomorphism*, whose kernel coincides with the maximal proper  $\preceq$ -convex subgroup of  $F$ . Since  $F'$  is a non Abelian simple group [6], this homomorphism factors through  $F/F' \sim \mathbb{Z}^2$ , where the last isomorphism is given by  $fF' \mapsto (\log_2(f'_+(0)), \log_2(f'_-(1)))$ . Hence, we may write (each representative of the class of)  $\tau$  in the form

$$\tau_{\preceq}(f) = a \log_2(f'_+(0)) + b \log_2(f'_-(1)).$$

A canonical representative is obtained by taking  $a, b$  so that  $a^2 + b^2 = 1$ . We will call this the *normalized Conrad homomorphism* associated to  $\preceq$ . In many cases, we will consider this homomorphism as defined on  $\mathbb{Z}^2 \sim F/F'$ , so that  $\tau_{\preceq}((m, n)) = am + bn$ , and we will identify  $\tau_{\preceq}$  to the ordered pair  $(a, b)$ .

Now let  $\preceq^*$  be a bi-ordering on  $F'$ . Let  $I_0 \subset ]0, 1[$  be a closed dyadic interval, and consider the induced bi-ordering on  $F \sim F_{I_0}$ , which we will just denote by  $\preceq$ . Let  $I \subset ]0, 1[$  be any other closed dyadic interval, and consider  $\tau_{\preceq, I}$  the corresponding normalized Conrad homomorphism defined on

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<sup>1</sup>The support of an element  $f \in F$  is the smallest closed set containing all the points which are not fixed by  $f$ .

$F_I$ . Since each  $F_I$  is conjugate to  $F_{I_0}$  by an element of  $F'$ , we have that  $\tau_{\preceq, I} = \tau_{\preceq, I_0}$  (as ordered pairs). Also, by definition,  $\tau_{\preceq} = \tau_{\preceq, I_0}$ .

**Lemma 5.1.1.** *If  $\tau_{\preceq}$  corresponds to the pair  $(a, b)$ , then either  $a=0$  or  $b=0$ .*

*Proof:* Assume by contradiction that  $a > 0$  and  $b > 0$  (all the other cases are analogous). Fix  $f \in F_{[1/2, 3/4]}$  such that  $f'_+(1/2) > 1$  and  $f'_-(3/4) < 1$ , and denote  $I_1 = [1/4, 3/4]$  and  $I_2 = [1/2, 7/8]$ . Viewing  $f$  as an element in  $F_{I_1} \sim F$ , we have

$$\tau_{\preceq, I_1}(f) = b \log_2(f'_-(3/4)) < 0.$$

Since Conrad's homomorphism is non-decreasing, this implies that  $f$  is negative with respect to the restriction of  $\preceq^*$  to  $F_{I_1}$ , and therefore  $f \prec^* id$ . Now viewing  $f$  as an element in  $F_{I_2} \sim F$ , we have

$$\tau_{\preceq, I_2}(f) = a \log_2(f'_+(1/2)) > 0,$$

which implies that  $f \succ^* id$ , thus giving a contradiction.  $\square$

We may now pass to the proof of Theorem 5.0.8. Indeed, assume that for the Conrad's homomorphism above one has  $a > 0$  and  $b = 0$ . We claim that  $\preceq^*$  then coincides with  $\preceq_{x-}^+$ . To show this, we need to show that a nontrivial element  $f \in F'$  is positive with respect to  $\preceq^*$  if and only if  $f'_+(x_f^-) > 1$ . Now such an  $f$  may be seen as an element in  $F_{[x_f^-, x_f^+]}$ , and viewed in this way Conrad's homomorphism yields

$$\tau_{\preceq, [x_f^-, x_f^+]}(f) = a \log_2(f'_+(x_f^-)).$$

Since  $a > 0$ , if  $f'_+(x_f^-) > 1$  then the right-hand member in this equality is positive. Since Conrad's homomorphism is non-decreasing, we have that  $f$  is positive with respect to  $\preceq^*$ . Analogously, if  $f'_+(x_f^-) < 1$  then  $f$  is negative with respect to  $\preceq^*$ .

Similar arguments show that the case  $a < 0, b = 0$  (resp.  $a = 0, b > 0$ , and  $a = 0, b < 0$ ) necessarily corresponds to the bi-ordering  $\preceq_{x-}^-$  (resp.  $\preceq_{x+}^-$ , and  $\preceq_{x+}^+$ ), which concludes the proof of Theorem 5.0.8.

**Question 5.1.2.** A bi-ordering whose positive cone is finitely generated as a normal semigroup is completely determined by finitely many inequalities (*i.e* it is isolated in the space of bi-orderings). This makes it natural to ask whether this is the case for the restrictions to  $F'$  of  $\preceq_{x-}^+$ ,  $\preceq_{x-}^-$ ,  $\preceq_{x+}^+$ , and  $\preceq_{x+}^-$ . A more sophisticated question is the existence of generators  $f, g$  of  $F'$  such that:

- $f'_+(x_f^-) > 1$ ,  $g'_+(x_g^-) > 1$ ,  $f'_-(x_f^+) < 1$ , and  $g'_-(x_g^+) > 1$ ,
- $F' \setminus \{id\}$  is the disjoint union of  $\langle \{f, g\} \rangle_N^+$  and  $\langle \{f^{-1}, g^{-1}\} \rangle_N^+$ ,
- $F' \setminus \{id\}$  is also the disjoint union of  $\langle \{f^{-1}, g\} \rangle_N^+$  and  $\langle \{f, g^{-1}\} \rangle_N^+$ .

A positive answer for the this question would immediately imply Theorem 5.0.8. Indeed, any bi-ordering  $\preceq$  on  $F'$  would be completely determined by the signs of  $f$  and  $g$ . For instance, if  $f \succ id$  and  $g \succ id$  then  $P_{\preceq}^+$  would necessarily contain  $\langle \{f, g\} \rangle_N^+$ , and by the second property above this would imply that  $\preceq$  coincides with  $\preceq_{x-}^+$ .

## 5.2 Bi-orderings on F

### 5.2.1 Isolated bi-orderings on F

Before classifying all bi-orderings on  $F$ , we will first give a proof of the fact that the eight elements in  $\mathcal{BO}_{Isol}(F)$  are isolated in  $\mathcal{BO}(F)$ . As in the case of  $F'$ , this proof strongly uses Conrad's homomorphism.

We just need to consider the cases of  $\preceq_{x^-}^+$  and  $\preceq_{0,x^-}^{+,-}$ . Indeed, all the other elements in  $\mathcal{BO}_{\text{Isol}}(F)$  are obtained from these by the action of the (finite Klein's) group generated by the involutions  $\preceq \mapsto \bar{\preceq}$  and  $\preceq \mapsto \preceq_\sigma$ ; see equation (5.1).

Let us first deal with  $\preceq_{x^-}^+$ , denoted  $\preceq$  for simplicity. Let  $(\preceq_k)$  be a sequence in  $\mathcal{BO}(F)$  converging to  $\preceq$ , and let  $\tau_k \sim (a_k, b_k)$  be the normalized Conrad's homomorphism for  $\preceq_k$  (so that  $\tau_k(m, n) = a_k m + b_k n$  and  $a_k^2 + b_k^2 = 1$ ).

**Claim 1.** For  $k$  large enough, one has  $b_k = 0$ .

Indeed, let  $f, g$  be two elements in  $F_{[1/2, 1]}$  which are positive with respect to  $\preceq$  and such that  $f'_-(1) = 1/2$  and  $g'_-(1) = 2$ . For  $k$  large enough, these elements must be positive also with respect to  $\preceq_k$ . Now notice that

$$\tau_k(f) = -b_k \quad \text{and} \quad \tau_k(g) = b_k.$$

Thus, if  $b_k \neq 0$ , then either  $f \prec_k id$  or  $g \prec_k id$ , which is a contradiction. Therefore,  $b_k = 0$  for  $k$  large enough.

Let us now consider the bi-ordering  $\preceq^*$  on  $F \sim F_{[1/2, 1]}$  obtained as the restriction of  $\preceq$ . Let  $\tau^* \sim (a^*, b^*)$  be the corresponding normalized Conrad's homomorphism.

**Claim 2.** One has  $b^* = 0$ .

Indeed, for the elements  $f, g$  in  $F_{[1/2, 1]}$  above, we have

$$\tau^*(f) = -b^* \quad \text{and} \quad \tau^*(g) = b^*.$$

If  $b^* \neq 0$ , this would imply that one of these elements is negative with respect to  $\preceq^*$ , and hence with respect to  $\preceq$ , which is a contradiction. Thus,  $b^* = 0$ .

Denote now by  $\preceq_k^*$  the restriction of  $\preceq_k$  to  $F_{[1/2, 1]}$ , and let  $\tau_k^* \sim (a_k^*, b_k^*)$  be the corresponding normalized Conrad's homomorphism.

**Claim 3.** For  $k$  large enough, one has  $b_k^* = 0$ .

Indeed, the sequence  $(\preceq_k^*)$  clearly converges to  $\preceq^*$ . Knowing also that  $b^* = 0$ , the proof of this claim is similar to that of Claim 1.

**Claim 4.** For  $k$  large enough, one has  $a_k > 0$  and  $a_k^* > 0$ .

Since Conrad's homomorphism is nontrivial, both  $a_k$  and  $a_k^*$  are nonzero. Take any  $f \in F$  such that  $f'_+(0) = 2$ . We have  $\tau_k(f) = a_k$ . Hence, if  $a_k < 0$ , then  $f \prec_k id$ , while  $f \succ id$ ... Analogously, if  $a_k^* < 0$ , then one would have  $g \prec_k id$  and  $g \succ id$  for any  $g \in F_{[1/2, 1]}$  satisfying  $g'(1/2) = 2$ .

**Claim 5.** If  $a_k$  and  $a_k^*$  are positive and  $b_k$  and  $b_k^*$  are zero, then  $\preceq_k$  coincides with  $\preceq$ .

Given  $f \in F$  such that  $f \succ id$ , we need to show that  $f$  is positive also with respect to  $\preceq_k$ . If  $x_f^- = 0$ , then  $f'_+(0) > 1$ , and since  $a_k > 0$ , this gives  $\tau_k(f) = a_k \log_2(f'_+(0)) > 0$ , and thus  $f \succ_k id$ .

If  $x_f^- \neq 0$ , then  $f'_+(x_f^-) > 1$ . In the case  $x_f = 1/2$ , since  $a_k^* > 0$ , we have that  $\tau_k^*(f) = a_k^* \log_2(f'_+(x_f^-)) > 0$ , and therefore one has  $f \succ_k id$ . In the case  $0 < x_f \neq 1/2$ , we can conjugate  $f$  by  $h \in F$  such that  $x_{hfh^{-1}} = 1/2$ . As before, we get  $\tau_k^*(hfh^{-1}) > 0$ , and therefore one still has  $f \succ_k id$ .

The proof for  $\preceq_{0,x^-}^{+,-}$  is similar to the above one. Indeed, Claims 1, 2, and 3 still hold. Concerning Claim 4, one now has that  $a_k > 0$  and  $a_k^* < 0$  for  $k$  large enough. Having this in mind, one easily concludes that  $\preceq_k$  coincides with  $\preceq_{0,x^-}^{+,-}$  for  $k$  very large.

### 5.2.2 Classifying all bi-orderings on F

To simplify, we will denote by  $\Lambda$  the union of  $\Lambda_{x-}^+$ ,  $\Lambda_{x-}^-$ ,  $\Lambda_{x+}^+$ , and  $\Lambda_{x+}^-$ . To prove our main result, fix a bi-ordering  $\preceq$  on F, and let  $\tau_{\preceq}: F \rightarrow (\mathbb{R}, +)$  be the corresponding normalized Conrad's homomorphism. Since  $\tau_{\preceq} \sim (a, b)$  is nontrivial and factors through  $\mathbb{Z}^2 \sim F/F'$ , there are two different cases to be considered.

**Case 1.** The image  $\tau_{\preceq}(\mathbb{Z}^2)$  has rank two.

This case appears when the quotient  $a/b$  is irrational. In this case,  $\preceq$  induces the bi-ordering of irrational type  $\preceq_{a/b}$  on  $\mathbb{Z}^2$  viewed as  $F/F'$ . Indeed, for each  $f \in F \setminus F'$  the value of  $\tau_{\preceq}(f)$  is nonzero, and hence it is positive if and only if  $f \succ id$ .

The kernel of  $\tau_{\preceq}$  coincides with  $F'$ . By Theorem 5.0.8, the restriction of  $\preceq$  to  $F'$  must coincide with one of the bi-orderings  $\preceq_{x-}^+$ ,  $\preceq_{x-}^-$ ,  $\preceq_{x+}^+$ , or  $\preceq_{x+}^-$ . Therefore,  $\preceq$  is contained in  $\Lambda$ , and the bi-ordering induced on the  $\mathbb{Z}^2$ -fiber is of irrational type.

**Case 2.** The image  $\tau_{\preceq}(\mathbb{Z}^2)$  has rank one.

This is the difficult case: it appears when either  $a/b$  is rational or  $b=0$ . There are two sub-cases.

*Subcase 1.* Either  $a=0$  or  $b=0$ .

Assume first that  $b=0$ . Denote by  $\preceq^*$  the bi-ordering induced on  $F_{[1/2,1]}$ , and let  $\tau_{\preceq^*} \sim (a^*, b^*)$  be its normalized Conrad's homomorphism. We claim that either  $a^*$  or  $b^*$  is equal to zero. Indeed, suppose for instance that  $a^* > 0$  and  $b^* > 0$  (all the other cases are analogous). Let  $m, n$  be integers such that  $n > 0$  and  $a^*m - b^*n > 0$ , and let  $f$  be an element in  $F_{[3/4,1]}$  such that  $f'_+(3/4) = 2^m$  and  $f'_-(1) = 2^{-n}$ . Then  $\tau_{\preceq^*}(f) = -b^*n < 0$ , and hence  $f \prec id$ . On the other hand, taking  $h \in F$  such that  $h(3/4) = 1/2$ , we get that  $h^{-1}fh \in F_{[1/2,1]}$ , and

$$\tau_{\preceq^*}(h^{-1}fh) = a^* \log_2((h^{-1}fh)'_+(1/2)) + b^* \log_2((h^{-1}fh)'_-(1)) = am - bn > 0.$$

But this implies that  $h^{-1}fh$ , and hence  $f$ , is positive with respect to  $\preceq$ , which is a contradiction.

(i) If  $a > 0$  and  $a^* > 0$ : We claim that  $\preceq$  coincides with  $\preceq_{x-}^+$  in this case. Indeed, let  $f \in F$  be an element which is positive with respect to  $\preceq_{x-}^+$ . We need to show that  $f \succ id$ . Now, since  $a > 0$ , if  $x_f^- = 0$  then

$$\tau_{\preceq}(f) = a \log_2(f'_+(0)) > 0,$$

and hence  $f \succ id$ . If  $x_f^- \neq 0$  then taking  $h \in F$  such that  $h(x_f^-) = 1/2$  we obtain that  $h^{-1}fh \in F_{[1/2,1]}$ , and

$$\tau_{\preceq^*}(h^{-1}fh) = a^* \log_2((h^{-1}fh)'(1/2)) = a^* \log_2(f'(x_f^-)).$$

Since  $a^* > 0$ , the value of the last expression is positive, which implies that  $h^{-1}fh$ , and hence  $f$ , is positive with respect to  $\preceq$ .

(ii) If  $a > 0$  and  $a^* < 0$ : Similar arguments to those of (i) above show that  $\preceq$  coincides with  $\preceq_{0,x-}^{+,-}$  in this case.

(iii) If  $a > 0$  and  $b^* > 0$ : We claim that  $\preceq$  belongs to  $\Lambda$ , and that the induced bi-ordering on the  $\mathbb{Z}^2$ -fiber is the lexicographic one. To show this, we first remark that if  $f \in F \setminus F'$  is positive, then either  $f'_+(0) > 1$ , or  $f'_+(0) = 1$  and  $f'_-(1) > 1$ . Indeed, if  $f'_+(0) \neq 1$ , then the value of  $\tau_{\preceq}(f) = a \log_2(f'_+(0)) \neq 0$  must be positive, since Conrad's homomorphism is non-decreasing. If  $f'_+(0) = 1$ , we take  $h \in F$  such that  $h(1/2) = x_f^-$ . Then  $h^{-1}fh$  belongs to  $F_{[1/2,1]}$ , and the value of

$$\tau_{\preceq^*}(h^{-1}fh) = b^* \log_2((h^{-1}fh)'_-(1)) = b^* \log_2(f'_-(1)) \neq 0$$



must be positive, since  $f$  (and hence  $h^{-1}fh$ ) is a positive element of  $F$ .

To show that  $\preceq$  induces a bi-ordering on  $\mathbb{Z}^2$ , we need to check that  $F'$  is  $\preceq$ -convex. Let  $g \in F'$  and  $h \in F$  be such that  $id \preceq h \preceq g$ . If  $h$  was not contained in  $F'$ , then  $hg^{-1}$  would be a negative element in  $F \setminus F'$ . But since

$$(hg^{-1})'_+(0) = h'_+(0) \quad \text{and} \quad (hg^{-1})'_-(1) = h'_-(1),$$

this would contradict the remark above. Therefore,  $h$  belongs to  $F'$ , which shows the  $\preceq$ -convexity of  $F'$ . Again, the remark above shows that the induced bi-ordering on  $\mathbb{Z}^2$  is the lexicographic one.

(iv) If  $a > 0$  and  $b^* < 0$ : As in (iii) above,  $\preceq$  belongs to  $\Lambda$ , and the induced bi-ordering  $\preceq_{\mathbb{Z}^2}$  on the  $\mathbb{Z}^2$ -fiber is the one for which  $(m, n) \succ_{\mathbb{Z}^2} (0, 0)$  if and only if either  $m > 0$ , or  $m = 0$  and  $n < 0$ .

(v) If  $a < 0$  and  $a^* > 0$ : As in (i) above,  $\preceq$  coincides with  $\preceq_{0,x-}^{-,+}$  in this case.

(vi) If  $a < 0$  and  $a^* < 0$ : As in (i) above,  $\preceq$  coincides with  $\preceq_{x-}^{-}$  in this case.

(vii) If  $a < 0$  and  $b^* > 0$ : As in (iii) above,  $\preceq$  belongs to  $\Lambda$ , and the induced bi-ordering  $\preceq_{\mathbb{Z}^2}$  on the  $\mathbb{Z}^2$ -fiber is the one for which  $(m, n) \succ_{\mathbb{Z}^2} (0, 0)$  if and only if either  $m < 0$ , or  $m = 0$  and  $n > 0$ .

(viii) If  $a < 0$  and  $b^* < 0$ : As in (iii) above,  $\preceq$  belongs to  $\Lambda$ , and the induced bi-ordering  $\preceq_{\mathbb{Z}^2}$  on the  $\mathbb{Z}^2$ -fiber is the one for which  $(m, n) \succ_{\mathbb{Z}^2} (0, 0)$  if and only if either  $m < 0$ , or  $m = 0$  and  $n < 0$ .

The case  $a = 0$  is analogous to the preceding one. Letting now  $\preceq^*$  be the restriction of  $\preceq$  to  $F_{[0,1/2]}$ , for the normalized Conrad's homomorphism  $\tau_{\preceq^*} \sim (a^*, b^*)$  one may check that either  $a^* = 0$  or  $b^* = 0$ .

Assume that  $b > 0$ . In the case  $b^* > 0$  (resp.  $b^* < 0$ ), the bi-ordering  $\preceq$  coincides with  $\preceq_{x+}^{-}$  (resp.  $\preceq_{1,x+}^{-,+}$ ). If  $a^* > 0$  (resp.  $a^* < 0$ ), then  $\preceq$  corresponds to a point in  $\Lambda$  whose induced bi-ordering  $\preceq_{\mathbb{Z}^2}$  on the  $\mathbb{Z}^2$ -fiber is the one for which  $(m, n) \succ_{\mathbb{Z}^2} (0, 0)$  if and only if either  $n > 0$ , or  $n = 0$  and  $m > 0$  (resp. either  $n > 0$ , or  $n = 0$  and  $m < 0$ ).

Assume now that  $b < 0$ . In the case  $b^* > 0$  (resp.  $b^* < 0$ ), the bi-ordering  $\preceq$  coincides with  $\preceq_{1,x+}^{+,-}$  (resp.  $\preceq_{x+}^{+}$ ). If  $a^* > 0$  (resp.  $a^* < 0$ ), then  $\preceq$  corresponds to a point in  $\Lambda$  whose induced bi-ordering  $\preceq_{\mathbb{Z}^2}$  on the  $\mathbb{Z}^2$ -fiber is the one for which  $(m, n) \succ_{\mathbb{Z}^2} (0, 0)$  if and only if either  $n < 0$ , or  $n = 0$  and  $m > 0$  (resp. either  $n < 0$ , or  $n = 0$  and  $m < 0$ ).

*Subcase 2.* Both  $a$  and  $b$  are nonzero.

The main issue here is to show that  $F'$  is necessarily  $\preceq$ -convex in  $F$ . Now since  $\ker(\tau_{\preceq})$  is already  $\preceq$ -convex in  $F$ , to prove this it suffices to show that  $F'$  is  $\preceq$ -convex in  $\ker(\tau_{\preceq})$ . Assume by contradiction that  $f$  is a positive element in  $\ker(\tau_{\preceq}) \setminus F'$  that is smaller than some  $h \in F'$ . Since both  $a, b$  are non zero, we have that  $f'_+(0) \neq 1$  and  $f'_-(1) \neq 1$ . Suppose first that  $\preceq$  restricted to  $F'$  coincides with either  $\preceq_{x-}^{+}$  or  $\preceq_{x-}^{-}$ , and denote by  $p$  the leftmost fixed point of  $f$  in  $]0, 1]$ . We claim that  $f$  is smaller than any positive element  $g \in F_{]0,p[}$ . Indeed, since  $\preceq$  coincides with either  $\preceq_{x-}^{+}$  or  $\preceq_{x-}^{-}$  on  $F'$ , the element  $f$  is smaller than any positive  $\bar{h} \in F_{]0,p[}$  such that  $x_h^+$  is to the left of  $x_h^-$ ; taking  $n \in \mathbb{Z}$  such that  $f^{-n}(x_h^-)$  is to the right of  $x_g^-$ , this gives  $f = f^{-n}ff^n \prec f^{-n}\bar{h}f^n \prec g$ .

Now take a positive element  $h_0 \in F_{]0,p[}$  such that for  $\bar{f} = h_0f$  there is no fixed point in  $]0, p[$  (it suffices to consider a positive  $h_0 \in F_{[\frac{p}{4}, \frac{3p}{4}]}$  whose graph is very close to the diagonal). Then  $id \prec \bar{f} \prec h_0g$  for every positive  $g \in F_{]0,p[}$ . The argument above then shows that  $\bar{f}$  is smaller than every positive element in  $F_{]0,p[}$ . In particular, since  $h_0 = \bar{f}f^{-1}$  is in  $F_{]0,p[}$  and is positive, this implies that  $\bar{f} \prec \bar{f}f^{-1}$ , and hence  $f \prec id$ , which is a contradiction.

If the restriction of  $\preceq$  to  $F'$  coincides with either  $\preceq_{x+}^{+}$  or  $\preceq_{x+}^{-}$ , one proceeds similarly but working on the interval  $[q, 1]$  instead of  $[0, p]$ , where  $q$  denotes the rightmost fixed point of  $f$  in  $[0, 1]$ . This concludes the proof of the  $\preceq$ -convexity of  $F'$ , and hence the proof of Theorem G.

**Remark 5.2.1.** Our arguments may be easily modified to show that the subgroup  $F_- = \{f \in F : f'_+(0) = 1\}$  has six different bi-orderings, namely (the restrictions of)  $\preceq_{x^-}^+$ ,  $\preceq_{x^-}^-$ ,  $\preceq_{x^+}^+$ ,  $\preceq_{x^+}^-$ ,  $\preceq_{1,x^+}^{+,+}$ , and  $\preceq_{1,x^+}^{-,+}$ . An analogous statement holds for  $F_+ = \{f \in F : f'_-(1) = 1\}$ . Finally, the group of piecewise-affine orientation-preserving dyadic homeomorphisms of the real line whose support is bounded from the right (resp. from the left) admits only two bi-orderings, namely (the natural analogues of)  $\preceq_{x^+}^+$  and  $\preceq_{x^+}^-$  (resp.  $\preceq_{x^-}^+$  and  $\preceq_{x^-}^-$ ); compare [12].

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